Prime Ideals of the Cartesian Product of Two Ordered Semigroups

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Abstract

An ordered semigroup is a semigroup \((S, \cdot)\) together with a partial order \(\leq\) on \(S\) such that \(x \leq y\) implies \(z \cdot x \leq z \cdot y\) and \(x \cdot z \leq y \cdot z\) for all \(x, y, z\) in \(S\). If \((S, \cdot, \leq_S)\) and \((T, \cdot, \leq_T)\) are two ordered semigroups, then the Cartesian product \(S \times T\) is a semigroup under the coordinatewise multiplication. Define a partial order \(\leq\) on \(S \times T\) by \((s_1, t_1) \leq (s_2, t_2)\) if and only if \(s_1 \leq_S s_2\) and \(t_1 \leq_T t_2\) for all \((s_1, t_1), (s_2, t_2) \in S \times T\). Then \(S \times T\) is an ordered semigroup. In this note, necessary and sufficient conditions of a subset of \(S \times T\) to be a prime ideal will be presented.

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1 Introduction

An ordered semigroup \([1]\) is a semigroup \((S, \cdot)\) with a partial order \(\leq\) on \(S\) such that \(x \leq y\) implies \(zx \leq zy\) and \(xz \leq yz\) for all \(x, y, z\) in \(S\). For a subset \(A\) of \(S\), let

\[ (A) = \{ x \in S \mid x \leq a \text{ for some } a \in A \}. \]

If \((S, \cdot, \leq)\) is an ordered semigroup, it is customary to write \(x \cdot y\) as \(xy\) for all \(x, y \in S\). The notion of ordered semigroup have been widely studied: see \([2], [3], [4], [5]\).

Let \((S, \cdot, \leq)\) be an ordered semigroup. A non-empty subset \(T\) of \(S\) is called a subsemigroup of \(S\) if \(xy \in T\) for all \(x, y \in T\).
Let \((S, \cdot, \leq)\) be an ordered semigroup. A nonempty subset \(I\) of \(S\) is called an (two-sided) ideal of \(S\) if the following holds.

(i) \(xy \in I\) and \(yx \in I\) for all \(x \in I, y \in S\).

(ii) For \(x \in I, y \in S, y \leq x\) implies \(y \in I\).

An ideal \(I\) of an ordered semigroup \((S, \cdot, \leq)\) is said to be prime if \(S \setminus I\) is a subsemigroup of \(S\). Note that if an ideal \(I\) of \(S\) is prime, then \(I \neq S\). To follows [6], we call the empty set a prime ideal.

Let \((S, \cdot, \leq_S)\) and \((T, \cdot, \leq_T)\) be two ordered semigroups. Under the coordinatewise multiplication, the Cartesian product \(S \times T\) of \(S\) and \(T\) forms a semigroup. Define a partial order \(\leq\) on \(S \times T\) by

\[
(s_1, t_1) \leq (s_2, t_2) :\iff s_1 \leq_S s_2, t_1 \leq_T t_2
\]

for all \((s_1, t_1), (s_2, t_2) \in S \times T\). Then \(S \times T\) is an ordered semigroup. As in [6], the purpose of this paper is to give necessary and sufficient conditions of a subset of \(S \times T\) to be a prime ideal.

\section{Main Results}

\textbf{Theorem 2.1} Let \((S, \cdot, \leq_S)\) and \((T, \cdot, \leq_T)\) be ordered semigroups. Then a subset \(L\) of \(S \times T\) is a prime ideal of \(S \times T\) if and only if there exist a prime ideal \(I\) of \(S\) and a prime ideal \(J\) of \(T\) such that \(L = (I \times T) \cup (S \times J)\).

\textbf{Proof.}\ The proof is a modification of the proof of the theorem in [6]. Assume that there exist a prime ideal \(I\) of \(S\) and a prime ideal \(J\) of \(T\) such that

\[
L = (I \times T) \cup (S \times J).
\]

If \(I = \emptyset\) and \(J = \emptyset\), then \(L = \emptyset\). Hence \(L\) is a prime ideal of \(S \times T\) by definition. Suppose that \(I \neq \emptyset\) or \(J \neq \emptyset\). Then \(L \neq \emptyset\). We shall show that \(L\) is a prime ideal of \(S \times T\). Let \((x, u) \in L\) and \((y, v) \in S \times T\). If \(x \in I\), then \(xy \in I\) and \(yx \in I\). Hence

\[
(x, u)(y, v) = (xy, uv) \in I \times T\quad \text{and}\quad (y, v)(x, u) = (yx, vu) \in I \times T.
\]

Thus \((x, u)(y, v), (y, v)(x, u) \in L\). Let \((x, u) \in L\) and \((y, v) \in S \times T\) be such that \((y, v) \leq (x, u)\). If \(x \in I\), then \(y \in I\) (since \(y \leq_S x\)). Thus \((y, v) \in I \times T\), and hence \((y, v) \in L\). Similarly, if \(u \in J\), then \((y, v) \in L\). Therefore, \(L\) is an ideal of \(S \times T\).

Next, we assert that \((S \times T) \setminus L\) is a subsemigroup of \(S \times T\). Since \(S \setminus I \neq \emptyset\) and \(T \setminus J \neq \emptyset\), \(S \setminus I\) and \(T \setminus J\) are semigroups of \(S\) and \(T\), respectively. We have
(S \times T) \setminus L = (S \setminus I) \times (T \setminus J) \neq \emptyset.

Then (S \setminus I) \times (T \setminus J) is a subsemigroup of S \times T. Hence L is a prime ideal of S \times T.

Conversely, assume that L is a prime ideal of S \times T. If L = \emptyset, then L = (\emptyset \times T) \cup (S \times \emptyset). Assume that L \neq \emptyset. Then, there exists (x, u) \in L. We assert that \{x\} \times T \subseteq L or S \times \{u\} \subseteq L. Suppose \{x\} \times T \not\subseteq L and S \times \{u\} \not\subseteq L. Then, there exists v \in T and y \in S such that (x, v) \not\in L and such that (y, u) \not\in L. We have

\]

Since \((x, v)(y, u)(x, v)(y, u) \in (S \times T) \setminus L\), \((xy, v)(x, u)(y, vu) \in (S \times T) \setminus L\).

Since \((x, u) \in L\), \((xy, v)(x, u)(y, vu) \in L\). A contradiction. Hence, \{x\} \times T \subseteq L or S \times \{u\} \subseteq L. Let

\[A = \{x \in S \mid \{x\} \times T \subseteq L\}\] and \[B = \{u \in T \mid S \times \{u\} \subseteq L\}\]

and let \(I = \{A\}\) and \(J = \{B\}\). Let \((x, u) \in L\). Then \{x\} \times T \subseteq L or S \times \{u\} \subseteq L.

This implies that \(x \in I\) or \(u \in J\). Hence \((x, u) \in (I \times T) \cup (S \times J)\). Thus \(L \subseteq (I \times T) \cup (S \times J)\). The reverse inclusion is clear. Hence

\[L = (I \times T) \cup (S \times J)\].

We shall show that \(I\) is a prime ideal of \(S\). That \(J\) is a prime ideal of \(T\) can be proved similarly. If \(I = \emptyset\), then \(I\) is a prime ideal of \(S\). Assume that \(I \neq \emptyset\). If \(I = S\), then \(L = S \times T\), a contradiction (since \(L\) is a prime ideal of \(S \times T\)). Hence \(S \setminus I \neq \emptyset\). Similarly, \(T \setminus J \neq \emptyset\). Let \(x, y \in S \setminus I\) and \(u \in T \setminus J\). Then

\[(x, u), (y, u) \in (S \setminus I) \times (T \setminus J)\].

Since \(L\) is prime, we have \((S \setminus I) \times (T \setminus J)\) is a subsemigroup of \(S \times T\). Since

\[(xy, u^2) = (x, u)(y, u) \in (S \setminus I) \times (T \setminus J)\]

we get \(xy \in S \setminus I\). Thus \(S \setminus I\) is a subsemigroup of \(S\). Let \(x \in I\), \(y \in S\) and \(u \in T \setminus J\). Since \(x \in I\), \((x, u) \in L\). Since \(L\) is an ideal of \(S \times T\), we have

\[(x, u)(y, u) = (xy, u^2) \in L\] and \((y, u)(x, u) = (yx, u^2) \in L\).

Since \(T \setminus J\) is a semigroup, \(u^2 \in T \setminus J\). Since

\[(xy, u^2), (yx, u^2) \in L\]

we obtain

\[(xy, u^2), (yx, u^2) \in I \times T\]
and hence \( xy, yx \in I \). It is clear that if \( x \in I \) and \( y \in S \) such that \( y \leq x \), then \( y \in I \). Therefore, \( I \) is a prime ideal of \( S \).

Suppose that \( S \) and \( T \) are semigroups. Then the Cartesian product \( S \times T \) is a semigroup under the coordinatewise multiplication. Define a partial order \( \leq_S \) on \( S \) by \( x \leq_S y \) if and only if \( x = y \) for all \( x, y \in S \). Then \( S \) forms an ordered semigroup. Similarly, \( T \) forms an ordered semigroup with a partial order \( \leq_T \) defined by \( x \leq_T y \) if and only if \( x = y \) for all \( x, y \in T \). Using Theorem 2.1, we have the following result proved in [6].

**Corollary 2.2** Let \( S \) and \( T \) be semigroups. Then a subset \( L \) of \( S \times T \) is a prime ideal of \( S \times T \) if and only if \( L = (I \times T) \cup (S \times J) \) for some prime ideals \( I \) and \( J \) of \( S \) and \( T \), respectively.

**References**


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