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Estimating Fractional Stochastic Volatility [1]

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Abstract

The aim of this paper is to find the best state estimation for fractional stochastic volatility from point process observations by a method of semimartingale approximation.

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1 Introduction

It is known that stochastic volatility models for options and derivatives were developed out of a need to modify the Black-Scholes model, which failed to effectively take the volatility in the price of underlying security into account. The Black-Scholes model assumed that the volatility of the underlying security was constant, while stochastic volatility models categorized the price of the underlying security as a random variable or more general, a stochastic process. In its turn, the dynamics of this stochastic process can be driven by some other process (commonly by a Brownian motion). Various models have been created for stochastic volatility such as Heston model, Constant Elasticity of

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Variance model (CEV model), SABR volatility model, GARCH model, 3/2-model, Chen model,. . .
In many situations, the dynamics of the volatility has a long-range effect, for instance changes of very strong bluechips in a stock market. One thinks of a kind of stochastic volatility perturbed by a process of long memory such as a fractional Brownian motion.
In this paper we consider the problem of state estimation of a stochastic volatility V_t given by the fractional stochastic differential equation

$$dV_t = -bV_t dt + \sigma dB_t^H,$$

where B_t^H is a fractional Brownian motion of Liouville form, from a point process observation Y_t.
Recall that a fractional Brownian motion W_t^H with Hurst index H is a centered Gaussian process such that its covariance function $R(s,t) = EW_s^H W_t^H$ is given by

$$R(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \text{ where } 0 < H < 1.$$

If $H = \frac{1}{2}, R(s,t) = \min(s,t)$ and W_t^H is the usual standard Brownian motion that is a Markov process.
For $H \neq \frac{1}{2}, W_t^H$ is neither a semimartingale nor a Markov process. It is a process of long memory.
According to B. Mandelbrot, W_t^H has the following representation

$$W_t^H = \frac{1}{\Gamma(1+\alpha)}[Z_t + \int_0^t (t-s)^\alpha dW_s],$$

where W is a standard Brownian motion, $\alpha = H - \frac{1}{2}$ and

$$Z_t = \int_{-\infty}^0 [(t-s)^\alpha - (-s)^\alpha] dW_s.$$

Since Z_t has absolutely continuous trajectories [1] one can see that the term $B_t^H := \int_0^t (t-s)^\alpha dW_s$ has long memory in the following sense (see [4]). If $\rho_n = E(B_1^H(B_{n+1}^H - B_n^H))$ then the series $\sum_{n=0}^\infty \rho_n$ is either divergent or convergent with very late rate. B_t^H is not a semimartingale as well and it is known as a fractional Brownian motion of Liouville form.
Every stochastic processes here are supposed to be adapted to a filtration $(\mathcal{F}_t)$ in a probability space $(\Omega, \mathcal{F}, P)$. Now we try to estimate the stochastic volatility V_t satisfying

$$dV_t = -bV_t dt + \sigma dB_t^H, \ 0 \leq t \leq T, \tag{1.1.1}$$

$$V_0 \sim N(\mu_0, \nu_0^2), b \text{ and } \sigma \text{ are positive constants}$$

from an observation given by a point process of intensity λ_t

$$Y_t = \int_0^t \lambda_s ds + M_t, \tag{1.1.2}$$

where λ_t is a progressive process such that $E\int_0^t \lambda_s^2 ds < \infty$, M_t is a $\mathcal{F}_t$- martingale and V_0, B_t^H and M_t are independent.
We will use an approximation approach given in [7] to solve this problem.

2 Approximation for the fractional process B_t^H

In this section we will make a brief recall on a principal result for L^2-approximation of the fractional Brownian motion (refer to [7]).
For every $\epsilon > 0$ we define

$$B_t^{H,\epsilon} = \int_0^t (t - s + \epsilon)^{H-\frac{1}{2}} dW_s, \tag{2.2.1}$$

with $H \neq \frac{1}{2}, 0 < H < 1, 0 \leq t \leq T$.

Lemma 2.1. *The process $(B_t^{H,\epsilon}, t \geq 0)$ is a semimartingale with respect to the natural filtration $\mathcal{F}_t^W = \sigma(W_s, s \leq t)$. In fact, $B_t^{H,\epsilon}$ can be expressed as*

$$B_t^{H,\epsilon} = \alpha \int_0^t \varphi_s^\epsilon ds + \epsilon^\alpha W_t, \tag{2.2.2}$$

where

$$\varphi_t^\epsilon = \int_0^t (t - u + \epsilon)^{\alpha-1} dW_u, \ \alpha = H - \frac{1}{2}, 0 < H < 1. \tag{2.2.3}$$

Theorem 2.1. (See[7]) *$B_t^{H,\epsilon}$ converges to B_t^H in $L^2(\Omega)$ when ϵ tends to 0. This convergence is uniform with respect to $t \in [0, T]$. Moreover we have $||B_t^{H,\epsilon} - B_t^H||_{L^2(\Omega)} \leq C(\alpha)\epsilon^{\alpha+\frac{1}{2}}$ where $C(\alpha)$ depends only on α.*

By this theorem we can see now any fractional stochastic dynamical system driven by a fractional Brownian notion can be approximated in L^2 by a system governed by a semimartingale B_t^ϵ, where the traditional stochastic calculus can be applied. Then the remaining work is to prove the convergence of approximate solutions to the exact solution. Notice that the exact solution of (1.1) can be defined as an adapted process V_t satifying $V_t = -b\int_0^t V_s ds + B_t^H, V_0 \sim N(\mu_0, \nu_0^2)$.
And our method of estimating stochastic volatility V_t is included in this context.

3 Approximate Fractional Stochastic Volatility

Instead of (1.1) and (1.2) we consider the following estimating problem:
The system dynamics for volatility is given by

$$dV_t^\epsilon = -bV_t^\epsilon dt + \sigma dB_t^{H,\epsilon},\ 0 \le t \le T \tag{3.3.1}$$

$$V_0^\epsilon = V_0 \sim N(\mu_0, \nu_0^2).$$

And the observation dynamics is

$$Y_t = \int_0^t \lambda_s ds + M_t, 0 \le t \le T. \tag{3.3.2}$$

By replacing $B_t^{H,\epsilon}$ in (3.1) by its expression in (2.2) we have

$$dV_t^\epsilon = [-bV_t^\epsilon + \sigma\alpha\varphi_t^\epsilon]dt + \epsilon^\alpha \sigma dW_t. \tag{3.3.3}$$

We can see that (3.3) can be splitted into two equations

$$dV_t^{\epsilon,1} = -bV_t^{\epsilon,1}dt + \epsilon^\alpha \sigma dW_t, \tag{3.3.4}$$

$$dV_t^{\epsilon,2} = [-bV_t^{\epsilon,2} + \sigma\alpha\varphi_t^\epsilon]dt. \tag{3.3.5}$$

Where (3.4) is a traditional stochastic Langevin equation and (3.5) is in fact an ordinary differential equation. And the solution V_t^ϵ of (3.3) should be $V_t^\epsilon = V_t^{\epsilon,1} + V_t^{\epsilon,2}$.
So the state estimation problem is splitted into two problems as well

$$E(V_t^\epsilon|\mathcal{F}_t^Y) = E(V_t^{\epsilon,1}|\mathcal{F}_t^Y) + E(V_t^{\epsilon,2}|\mathcal{F}_t^Y).$$

Notation:

(i) The state estimation of a process X_t based on the information $\mathcal{F}_t^Y$ given by an observation process Y_t will be denoted by $\hat{X}_t$ or $\pi_t(X)$. More general, the notation $\pi_t(f)$ stands for the state estimation $E(f(X_t)|\mathcal{F}_t^Y)$, where f is a bounded and continuous function, $f \in C_b(\mathbb{R})$. Recall that $\mathcal{F}_t^Y = \sigma(Y_s, s \le t)$ is the σ-algebra generated by all random variable Y_s with $0 \le s \le t$.

(ii) Suppose that the probability P is obtained from a probability Q by an absolutely continuous change of measure $Q \mapsto P$ such that $n_t = Y_t - t$

is a $(Q, \mathcal{F}_t^Y)$- martingale.
A Bayes formula gives us

$$E_P(X_t|\mathcal{F}_t^Y) = \frac{E_Q(X_t L_t|\mathcal{F}_t^Y)}{E_Q(L_t)},$$

where

$$L_t = E[\frac{dP}{dQ}|\mathcal{F}_t^Y] = \prod_{0\leq s\leq t} \lambda_s \triangle Y_s \exp \int_0^t (1-\lambda_s)ds.$$

Denote by $\sigma(X_t)$ or $\tilde{X}_t$ the unnormalized state estimation of X_t, given by

$$\tilde{X}_t = \sigma(X_t) = E_Q(L_t X_t|\mathcal{F}_t^Y),$$

or more general

$$\sigma(f) = E_Q(L_t f(X_t)|\mathcal{F}_t^Y), \text{ for } f \in C_b(\mathbb{R}).$$

4 Estimating $V_t^{\epsilon,1}$

Theorem 4.1. *(i) The state estimation $\pi_t(f) = E[f(V_t^{\epsilon,1})|\mathcal{F}_t^Y]$ for the system dynamics (3.4) with the point process observation (3.2) is given by*

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s[-bV_t^{\epsilon,1} f'(V_t^{\epsilon,1}) + \frac{(\epsilon^\alpha \sigma)^2}{2b} f''(V_t^{\epsilon,1})]ds$$

$$+ \int_0^t \pi_s^{-1}(\lambda)[\pi_{s-}(\lambda f) - \pi_s(f)\pi_{s-}(\lambda)]dm_s, \tag{4.4.1}$$

where $m_t = Y_t - \int_0^t \lambda_s ds$, $f \in C_b^2(\mathbb{R})$.

(ii) The unnormalized state estimation $\sigma_t(f) = E_Q[L_t f(V_t^{\epsilon,1})|\mathcal{F}_t^Y]$ is given by

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s[-bV_t^{\epsilon,1} f'(V_t^{\epsilon,1}) + \frac{(\epsilon^\alpha \sigma)^2}{2b} f''(V_t^{\epsilon,1})]ds$$

$$+ \int_0^t [\sigma_{s-}(\lambda f) - \sigma_s(f)]d\mu_s, \tag{4.4.2}$$

where $\mu_t = Y_t - t$, $f \in C_b^2(\mathbb{R})$.

Proof. It is known that the solution $V_t^{\epsilon,1}$ is in fact an Ornstein-Uhlenbeck process that is a Markov process and its semigroup is defined by a family $(P_t, t \geq 0)$ of operations on bounded Borelian function f.

$$(P_t f)(x) = \int_{\mathbb{R}} f[e^{-bt}x + \frac{(\epsilon^\alpha \sigma)^2}{2b}\sqrt{1 - e^{-2bt}y}]\mu(dy), \tag{4.4.3}$$

where μ is a standard Gaussian measure on $\mathbb{R}$

$$\mu(dy) = \frac{1}{\sqrt{2\pi}} \exp\,(\frac{-y^2}{2})dy.$$

It is obvious that $\lim\limits_{t \searrow 0}(P_t f)(x) = f(x)$ then $V_t^{\epsilon,1}$ is a Feller process. The corresponding infinitesimal operator A_t is given by

$$(A_t f)(x) = \lim_{t \to 0} \frac{1}{t}(P_t f - f)(x) = -bxf^{'}(x) + \frac{(\epsilon^\alpha \sigma)^2}{2b} f^{''}(x). \tag{4.4.4}$$

According to a result from [5], the state estimation $\pi_t(f)$ for a Fellerian system process from a point process observation $Y_t = \int_0^t \lambda_s ds + M_t$ is given by

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(Af)ds + \int_0^t \pi_s^{-1}(\lambda)[\pi_{s-}(\lambda f) - \pi_s(f)\pi_{s-}(\lambda)]dm_s,$$

where $m_t = Y_t - \int_0^t \pi_s(\lambda)ds$, provided $\pi_t^{-1} \neq 0$ and the unnormalized state estimation $\sigma_t(f)$ is given by

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(Af)ds + \int_0^t [\sigma_{s-}(\lambda f) - \sigma_s(f)]d\mu_s,$$

where $\mu_t = Y_t - t$.
By replacing Af in these equations by its expression (4.4) we get equations (4.1) and (4.2) as required. □

Corollary 4.1. *By taking $f = I$ (identification function) we have*

$$\widehat{V_t^{\epsilon,1}} = \widehat{V_0^{\epsilon,1}} - \int_0^t b\widehat{V_s^{\epsilon,1}}ds + \int_0^t \frac{1}{\hat{\lambda}_s}[\widehat{\lambda_s V_s^{\epsilon,1}} - \hat{\lambda}_{s-}\widehat{V_s^{\epsilon,1}}](dY_s - \hat{\lambda}_s ds). \tag{4.4.5}$$

$$\widetilde{V_t^{\epsilon,1}} := \sigma_t(I) = \widetilde{V_0^{\epsilon,1}} - \int_0^t b\widetilde{V_s^{\epsilon,1}}ds + \int_0^t [\tilde{\lambda}_{s-} - \widetilde{V_s^{\epsilon,1}}](dY_s - ds). \tag{4.4.6}$$

5 Estimating $V_t^{\epsilon,2}$ and V_t^{ϵ}.

It is clear from (3.5) that

$$\widehat{V_t^{\epsilon,2}} = \widehat{V_0^{\epsilon,2}} + \int_0^t [-b\widehat{V_s^{\epsilon,2}} + \sigma\alpha\hat{\varphi}_s^{\epsilon}]ds \tag{5.5.1}$$

and

$$\widetilde{V_t^{\epsilon,2}} = \widetilde{V_0^{\epsilon,2}} + \int_0^t [-b\widetilde{V_s^{\epsilon,2}} + \sigma\alpha\tilde{\varphi}_s^{\epsilon}]ds. \tag{5.5.2}$$

Remark 5.1. These equations have explicit solutions as follows:

$$\widehat{V_t^{\epsilon,2}} = \widehat{V_0^{\epsilon,2}}e^{-bt} - \sigma\alpha\int_0^t e^{-b(t-s)}\hat{\varphi}_s^{\epsilon}ds \tag{5.5.3}$$

and

$$\widetilde{V_t^{\epsilon,2}} = \widetilde{V_0^{\epsilon,2}}e^{-bt} - \sigma\alpha\int_0^t e^{-b(t-s)}\tilde{\varphi}_s^{\epsilon}ds. \tag{5.5.4}$$

Combining (4.5), (4.6) and (5.1), (5.2) we get

Corollary 5.1.

$$\widehat{V_t^{\epsilon}} = \widehat{V_t^{\epsilon,1}} + \widehat{V_t^{\epsilon,2}} \tag{5.5.5}$$

$$= \widehat{V_0^{\epsilon}} - \int_0^t b\widehat{V_s^{\epsilon}}ds + \int_0^t \sigma\alpha\widehat{\varphi_s^{\epsilon}}ds + \int_0^t \frac{1}{\widehat{\lambda_s}}[\widehat{\lambda_s V_s^{\epsilon,1}} - \hat{\lambda}_{s-}\widehat{V_s^{\epsilon,1}}](dY_s - \hat{\lambda}_s ds) \tag{5.5.6}$$

$$\widetilde{V_t^{\epsilon}} = \widetilde{V_t^{\epsilon,1}} + \widetilde{V_t^{\epsilon,2}} \tag{5.5.7}$$

$$= \widetilde{V_0^{\epsilon}} - b\int_0^t \widetilde{V_s^{\epsilon}}ds + \sigma\alpha\int_0^t \widetilde{\varphi_s^{\epsilon}}ds + \int_0^t [\tilde{\lambda}_{s-} - \widetilde{V_s^{\epsilon,1}}](dY_s - ds), \tag{5.5.8}$$

where we assume that $\widehat{V_0^{\epsilon,1}} + \widehat{V_0^{\epsilon,2}} = \widehat{V_0^{\epsilon}}$ *and* $\widetilde{V_0^{\epsilon,1}} + \widetilde{V_0^{\epsilon,2}} = \widetilde{V_0^{\epsilon}}$.

6 Convergence of V_t^{ϵ} to the solution V_t

Assume that V_t and V_t^{ϵ} are solutions of (1.1) and (3.1), as in [5] we can establish

Proposition 6.1. *V_t^{ϵ} converges to V_t in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$.*

Proof. We have

$$V_t - V_t^\epsilon = -b\int_0^t (V_s - V_s^\epsilon)ds + \sigma(B_t^H - B_t^{H,\epsilon}) \tag{6.6.1}$$

so

$$||V_t - V_t^\epsilon|| \leq ||b\int_0^t (V_s - V_s^\epsilon)ds|| + \sigma||B_t^H - B_t^{H,\epsilon}|| \tag{6.6.2}$$

It follows from (2.1) that

$$||V_t - V_t^\epsilon|| \leq b\int_0^t ||V_s - V_s^\epsilon||ds + \sigma C(\alpha)\epsilon^{\alpha+\frac{1}{2}}, \tag{6.6.3}$$

where $||.||$ stands for the norm in $L^2(\Omega)$.
Applying Gronwall's Lemma to (6.3) we will get

$$||V_t - V_t^\epsilon|| \leq \sigma C(\alpha)\epsilon^{\alpha+\frac{1}{2}}e^{bt}. \tag{6.6.4}$$

And we see that

$$\sup_{0\leq t\leq T} ||V_t - V_t^\epsilon|| \leq \sigma C(\alpha)\epsilon^{\alpha+1}e^{bT}. \tag{6.6.5}$$

So $V_t^\epsilon \longmapsto V_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0,T]$. □

7 Estimating volatility V_t

Proposition 7.1. *The state estimation $\hat{V}_t$ of the stochastic volatility V_t is the L^2-limit of $\hat{V}_t^\epsilon = \widehat{V_t^{\epsilon,1}} + \widehat{V_t^{\epsilon,2}}$ as $\epsilon \mapsto 0$ and the convergence is uniform with respect to $t \in [0,T]$.*

Proof. The assertion of this proposition follows from a result on the convergence of condition expectation (refer, for instance, to [3]). If $V_t^\epsilon \longmapsto^{L^2} V_t \in L^2$ as $\epsilon \mapsto 0$ then $E(V_t^\epsilon|\mathcal{F}_t^Y) \longmapsto E(V_t|\mathcal{F}_t^Y)$ as $\epsilon \mapsto 0$. And the uniform convergence is deduced from the estimation (6.5). □

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