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# Sums of Powers of Integers Divisible by Three 

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#### Abstract

A number of sequences based on sums of powers of integers divisible by three is presented. This approach provides a simple derivation of some well known sequences, as well as the construction of many new sequences.


The sums of powers of integers have been the subject of significant research interest over the years. This is because of commonly encountered combinatorial expressions such as

$$
\begin{equation*}
\sum_{i=1}^{n} i=1+2+3+\ldots+n=\frac{n(n+1)}{2} \tag{1}
\end{equation*}
$$

for the sum of the first $n$ natural numbers and

$$
\begin{equation*}
\sum_{i=1}^{n} i^{3}=1^{3}+2^{3}+3^{3}+\ldots+n^{3}=\left(\sum_{i=1}^{n} i\right)^{2} \tag{2}
\end{equation*}
$$

In this paper we examine the sums themselves, and in particular the integer sequences that result. Consider the sum of the first $n m$-th powers

$$
\begin{equation*}
\sum_{i=1}^{n} i^{m} \tag{3}
\end{equation*}
$$

For $n=4$, the values are

$$
\begin{gather*}
10,30,100,354,1300,4890,18700,72354,282340 \\
1108650,4373500,17312754, \ldots \tag{4}
\end{gather*}
$$

This is sequence A103438 in the online encyclopedia of integer sequences maintained by Sloane [1]. Sequences with other values of $n$ and $m$ were considered in [2]. An interesting feature of (4) is that the ones digit is 0 when $m \neq 4$.

In this paper, we generalize (3) to integers divisible by three, giving

$$
\begin{equation*}
\sum_{i=1}^{n}(3 i)^{m} \tag{5}
\end{equation*}
$$

For $n=1$, we have $3^{m}$. The corresponding sequence, $3,9,27,81,243,729,2187$, $\ldots$ has a units digit which repeats in a sequence $3,9,7,1$. For $n=2$ to 10 , the values are

| 9, | 45, | 243, | 1377, | 8019, | $\ldots$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 18, | 126, | 972, | 7938, | 67068, | $\ldots$ |
| 30, | 270, | 2700, | 28674, | 315900, | $\ldots$ |
| 45, | 495, | 6075, | 79299, | 1075275, | $\ldots$ |
| 63, | 819, | 11907, | 184275, | 2964843, | $\ldots$ |
| 84, | 1260, | 21168, | 378756, | 7048944, | $\ldots$ |
| 108, | 1836, | 34992, | 710532, | 15011568, | $\ldots$ |
| 135, | 2565, | 54675, | 1241973, | 29360475, | $\ldots$ |
| 165, | 3465, | 81675, | 2051973, | 53660475, | $\ldots$ |

The first two sequences in (6) are A063376 and A074533, respectively [1]. The other sequences are new. Comparing (4) and the third row in (6), one notices a similarity in the units digit, namely

$$
\begin{equation*}
10 \mid \sum_{i=1}^{4} i^{m} \Longleftrightarrow m \neq 0 \bmod 4 \tag{7}
\end{equation*}
$$

In order to show this more clearly, the values of (5) for $n=1$ to 10 (number of terms) and $m=1$ to 12 (power) reduced modulo 10 are given in Table 1 (of course the last two columns are identical because we are adding a power of 10).

## TABLE 1

| power | number of terms $n$ |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 3 | 9 | 8 | 0 | 5 | 3 | 4 | 8 | 5 | 5 |
| 2 | 9 | 5 | 6 | 0 | 5 | 9 | 0 | 6 | 5 | 5 |
| 3 | 7 | 3 | 2 | 0 | 5 | 7 | 8 | 2 | 5 | 5 |
| 4 | 1 | 7 | 8 | 4 | 9 | 5 | 6 | 2 | 3 | 3 |
| 5 | 3 | 9 | 8 | 0 | 5 | 3 | 4 | 8 | 5 | 5 |
| 6 | 9 | 5 | 6 | 0 | 5 | 9 | 0 | 6 | 5 | 5 |
| 7 | 7 | 3 | 2 | 0 | 5 | 7 | 8 | 2 | 5 | 5 |
| 8 | 1 | 7 | 8 | 4 | 9 | 5 | 6 | 2 | 3 | 3 |
| 9 | 3 | 9 | 8 | 0 | 5 | 3 | 4 | 8 | 5 | 5 |
| 10 | 9 | 5 | 6 | 0 | 5 | 9 | 0 | 6 | 5 | 5 |
| 11 | 7 | 3 | 2 | 0 | 5 | 7 | 8 | 2 | 5 | 5 |
| 12 | 1 | 7 | 8 | 4 | 9 | 5 | 6 | 2 | 3 | 3 |

From Table 1, it appears that for $n=4,5,9$ and 10

$$
5 \mid \sum_{i=1}^{n} i^{m} \Longleftrightarrow m \neq 0 \bmod 4
$$

To prove this result, one could consider expressions such as

$$
\begin{equation*}
\sum_{i=1}^{n} 3 i=3+6+9+\ldots+3 n=\frac{3}{2} n(n+1) \tag{8}
\end{equation*}
$$

and the following for $m=2,3$, and 4

$$
\begin{align*}
\sum_{i=1}^{n}(3 i)^{2} & =\frac{3}{2} n(n+1)(2 n+1)  \tag{9}\\
\sum_{i=1}^{n}(3 i)^{3} & =\frac{27}{4} n^{2}(n+1)^{2}  \tag{10}\\
\sum_{i=1}^{n}(3 i)^{4} & =\frac{27}{10} n(n+1)(2 n+1)\left(3 n^{2}+3 n-1\right) \tag{11}
\end{align*}
$$

respectively. Fortunately, there is a much simpler way.
Consider the residues modulo 10 of the powers of the integers that are a multiple of 3: $3,6,9,12,15,18,21,24,27$ and 30 , which are given in Table 2.

## Table 2

|  | integer |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| power | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 1 | 3 | 6 | 9 | 2 | 5 | 8 | 1 | 4 | 7 | 0 |
| 2 | 9 | 6 | 1 | 4 | 5 | 4 | 1 | 6 | 9 | 0 |
| 3 | 7 | 6 | 9 | 8 | 5 | 2 | 1 | 4 | 3 | 0 |
| 4 | 1 | 6 | 1 | 6 | 5 | 6 | 1 | 6 | 1 | 0 |
| 5 | 3 | 6 | 9 | 4 | 5 | 8 | 1 | 4 | 7 | 0 |
| 6 | 9 | 6 | 1 | 8 | 5 | 4 | 1 | 6 | 9 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

The periods of these residues are:

$$
\begin{array}{cl}
6,15,21,30 & \text { period 1 } \\
9,24 & \text { period 2 } \\
3,12,18,27 & \text { period 4 }
\end{array}
$$

which are all factors of $\phi(10)=4$. These values show that the periods of the units digits of (5) must be 4. Determining the units digits for the sums in the table can simply be done by summing the first $n$ columns of Table 2 (taking

Table 3

| power | number of terms $n$ |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\cdots$ |  |
| 1 | 3 | 9 | 8 | 0 | 5 | 3 | 4 | 8 | 5 | $\cdots$ |  |
| 2 | 9 | 5 | 6 | 0 | 5 | 9 | 0 | 6 | 5 | $\cdots$ |  |
| 3 | 7 | 3 | 2 | 0 | 5 | 7 | 8 | 2 | 5 | $\cdots$ |  |
| 4 | 1 | 7 | 8 | 4 | 9 | 5 | 6 | 2 | 3 | $\cdots$ |  |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |  |  |

columns modulo 10), which are given in Table 3. Since Table 3 will repeat for powers greater than 4 , the proof of the above result is complete.

For other residues, it is a simple matter to form the tables and determine how the sequences of units digits repeat. For example, taking the values of (5) modulo 3 trivially gives 0 , but modulo 4 gives the values shown in Table 4. Note that in this case the pairs of columns 1 and 2,3 and 4,5 and 6 , are identical except for the first element. A simple proof of this requires only the

Table 4

| power | number of terms $n$ |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |  |  |  |  |
| 1 | 3 | 1 | 2 | 2 | 1 | 3 | 0 | 0 | 3 | 1 |  |  |  |  |
| 2 | 1 | 1 | 2 | 2 | 3 | 3 | 0 | 0 | 1 | 1 |  |  |  |  |
| 3 | 3 | 3 | 0 | 0 | 3 | 3 | 0 | 0 | 3 | 3 |  |  |  |  |
| 4 | 1 | 1 | 2 | 2 | 3 | 3 | 0 | 0 | 1 | 1 |  |  |  |  |
| 5 | 3 | 3 | 0 | 0 | 3 | 3 | 0 | 0 | 3 | 1 |  |  |  |  |
| 6 | 1 | 1 | 2 | 2 | 3 | 3 | 0 | 0 | 1 | 1 |  |  |  |  |
| 7 | 3 | 3 | 0 | 0 | 3 | 3 | 0 | 0 | 3 | 3 |  |  |  |  |
| 8 | 1 | 1 | 2 | 2 | 3 | 3 | 0 | 0 | 1 | 1 |  |  |  |  |
| 9 | 3 | 3 | 0 | 0 | 3 | 3 | 0 | 0 | 3 | 1 |  |  |  |  |
| 10 | 1 | 1 | 2 | 2 | 3 | 3 | 0 | 0 | 1 | 3 |  |  |  |  |
| 11 | 3 | 3 | 0 | 0 | 3 | 3 | 0 | 0 | 3 | 1 |  |  |  |  |
| 12 | 1 | 1 | 2 | 2 | 3 | 3 | 0 | 0 | 1 | 1 |  |  |  |  |

residues modulo 4 of the powers of the integers a multiple of 3 , which are given in Table 5. This shows that every second column is zero except for the first element. The reason for this is that the first element contains 2 as a factor, while all others have $2^{m}$ as a factor, $m>1$, so the results modulo 4 are zero. The periods of the residues are either 1 or 2 , which is expected since $\phi(4)=2$.

TABLE 5

|  | integer |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| power | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 1 | 3 | 6 | 1 | 0 | 3 | 2 | 1 | 0 | 3 | 2 |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 3 | 0 | 1 | 0 | 3 | 0 | 1 | 0 | 3 | 0 |
| 4 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 5 | 3 | 0 | 1 | 0 | 3 | 0 | 1 | 0 | 3 | 0 |
| 6 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

In terms of divisibility, Table 4 shows that for $n=7$ and 8

$$
4 \mid \sum_{i=1}^{n}(3 i)^{m}
$$

Another interesting case is the residues modulo 5, which are given in Table 6 for $n$ up to 15 . Note that columns $4,5,9,10,14$ and 15 contain mostly

TABLE 6

| power |  |  |  |  |  |  |  | ber | of | term | ns $n$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ |  |  | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 1 |  | 4 | 43 | 0 | 0 | 3 | 4 | 3 | 0 | 0 | 3 | 4 | 3 | 0 | 0 |
| 2 |  | 0 | 1 | 0 | 0 | 4 | 0 | 1 | 0 | 0 | 4 | 0 | 1 | 0 | 0 |
| 3 |  | 3 | 32 | 0 | 0 | 2 | 3 | 2 | 0 | 0 | 2 | 3 | 2 | 0 | 0 |
| 4 |  | 2 | 23 | 4 | 4 | 0 | 1 | 2 | 3 | 3 | 4 | 0 | 1 | 2 | 2 |
| 5 |  | , | 43 | 0 | 0 | 3 | 4 | 3 | 0 | 0 | 3 | 4 | 3 | 0 | 0 |
| 6 |  | 0 | 1 | 0 | 0 | 4 | 0 | 1 | 0 | 0 | 4 | 0 | 1 | 0 | 0 |
| 7 |  | 3 | 32 | 0 | 0 | 2 | 3 | 2 | 0 | 0 | 2 | 3 | 2 | 0 | 0 |
| 8 | 1 | 2 | 23 | 4 | 4 | 0 | 1 | 2 | 3 | 3 | 4 | 0 | 1 | 2 | 2 |
| 9 |  |  | 43 | 0 | 0 | 3 | 4 | 3 | 0 | 0 | 3 | 4 | 3 | 0 | 0 |
| 10 |  | 0 | ) 1 | 0 | 0 | 4 | 0 | 1 | 0 | 0 | 4 | 0 | 1 | 0 | 0 |
| 11 |  | 3 | 32 | 0 | 0 | 2 | 3 | 2 | 0 | 0 | 2 | 3 | 2 | 0 | 0 |
| 12 |  | 2 | 23 | 4 | 4 | 0 | 1 | 2 | 3 | 3 | 4 | 0 | 1 | 2 | 2 |

zeros. From this table it is obvious that for $n=5 r$ or $5 r-1$

$$
5 \mid \sum_{i=1}^{n}(3 i)^{m} \Longleftrightarrow m \neq 0 \bmod 4
$$

Table 7

|  | integer |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| power | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 1 | 3 | 1 | 4 | 2 | 0 | 3 | 1 | 4 | 2 | 0 |
| 2 | 4 | 1 | 1 | 4 | 0 | 4 | 1 | 1 | 4 | 0 |
| 3 | 2 | 1 | 4 | 3 | 0 | 2 | 1 | 4 | 3 | 0 |
| 4 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 |
| 5 | 3 | 1 | 4 | 2 | 0 | 3 | 1 | 4 | 2 | 0 |
| 6 | 4 | 1 | 1 | 4 | 0 | 4 | 1 | 1 | 4 | 0 |
| 7 | 2 | 1 | 4 | 3 | 0 | 2 | 1 | 4 | 3 | 0 |
| 8 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

This is confirmed by the results in Table 7, which shows the residues modulo 5 of the powers of the integers a multiple of 3 , i.e., $(3 i)^{m} \bmod 5$. This table shows that for $i=0$ or 2 modulo 5 , the period is 1 , for $i=3$ modulo 5 , the period is 2 , and otherwise the period is 4 . These values are all factors of $\phi(5)=4$.

It is left to the reader to determine the results modulo other values, in particular, how does Table 4 differ from the values taken modulo 7? The period of the integer powers will be a factor of $\phi(7)=6$.

It is much more difficult to determine the divisibility when the values of (5) are considered with $m$ fixed. However, (9) shows that

$$
5 \mid \sum_{i=1}^{n}(3 i)^{2}
$$

when $n=5 r, 5 r-1$ or $5 r-3, r \geq 1$, and

$$
4 \mid \sum_{i=1}^{n}(3 i)^{2},
$$

only when $n=8 r$ or $8 r-1$. In addition, (10) shows that

$$
5 \mid \sum_{i=1}^{n}(3 i)^{3},
$$

when $n=5 r$ or $5 r-1$, and

$$
4 \mid \sum_{i=1}^{n}(3 i)^{3},
$$

when $n=4 r$ or $4 r-1$. The divisibility by other integers is easily established.

Many other divisibility identities can be established for fixed $n$ or $m$ simply by looking at the sums modulo a number, or by examining the closed form expressions for

$$
\sum_{i=1}^{n}(3 i)^{m}
$$

for fixed $m$. In addition, one could consider

$$
\sum_{i=1}^{n}(p i)^{m}
$$

for $p$ greater than 3 .
Returning to sequences, one can consider the columns in (6). The first column is given by (8), and is sequence A045943. The sequences from the other columns are new.

## References

[1] N.J.A. Sloane, On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/index.html.
[2] T.A. Gulliver, Divisibility of sums of powers of integers, Int. Math. J., 3 (2003), 699-704.

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