A Note on Amdeberhan-Moll’s Conjecture

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Abstract
Recently, Amdeberhan and Moll gave a conjecture on ASM sequences. In this note, we show that this conjecture is not true and we give a new conjecture.

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1 Introduction
Recall that a plane partition is a two-dimensional array of integers $\pi_{i,j}$ that are weakly decreasing in rows and columns and that add up to a given number $\pi$. In other words, $\pi_{i,j} \geq \pi_{i+1,j}$, $\pi_{i,j} \geq \pi_{i,j+1}$ and $\sum \pi_{i,j} = \pi$. These partitions are often represented by solid Young diagrams in 3-dimensions. Let $PP_n$ denote the number of plane partitions in the $n-$cube. It has the formula

$$PP_n = \prod_{i,j,k=1}^{n} \frac{i+j+k-1}{i+j+k-2}. \quad (1)$$

The set of plane partitions which are invariant under exchange of the $i$ and $j$ axes are called symmetric plane partitions. The number of such partitions whose solid Young diagrams fit inside an $n-$cube is denoted by $SPP_n$, which is simplified to

$$SPP_n = \prod_{i,j=1}^{n} \frac{i+j+n-1}{i+j+i-2} = \prod_{j=1}^{n} \prod_{i=j}^{n} \frac{i+j+n-1}{i+j-1}. \quad (2)$$

A plane partition invariant under any permutations of the three axes is called totally symmetric plane partitions (TSPP), Stembridge [3] proved that the number of TSPP in an $n-$cube is given by

$$TSPP_n = \prod_{j=1}^{n} \prod_{i=j}^{n} \frac{i+j+n-1}{i+j+i-2}. \quad (3)$$
A totally symmetric self-complementary plane partition (TSSCPP) is a plane partition which is invariant under permutation of the three axes and which is equal to its complement (i.e., the collection of cubes that are in a given box but do not belong to the solid Young diagram). Such partitions only fit in an even-dimensional box. The number of plane partitions inside a $2n \times 2n \times 2n$ box that are both totally symmetric and self-complementary (TSSCPP) is given by

$$TSSCPP_{2n} = \prod_{1 \leq i \leq j \leq n} \frac{i+j+n-1}{i+j+i-1}. \quad (4)$$

An alternating sign matrix (ASM) is a matrix of 0s, 1s, and $-1$s in which the entries in each row or column sum to 1 and the nonzero entries in each row and column alternate in sign. Let $A_n$ denote the number of ASM of size $n$ in this note. Zeilberger [5] completely proved the conjecture that

$$A_n = TSSCPP_{2n}. \quad (5)$$

For more details, see Bressoud’s book [2].

Recently, some arithmetic properties of $A_n$ are discovered by Sun and Moll [4]. Amdeberhan and Moll [1] studied the 2-adic valuations of the numbers of plane partition (PP), symmetric plane partition (SPP), totally symmetric plane partitions (TSPP), totally symmetric self-complementary plane partitions (TSSCPP) and alternating sign matrix (ASM). Given a sequence of positive numbers \( \{a_n\}_{n=0}^{\infty} \), Amdeberhan and Moll [1] gave the following definition

$$a_{n+1}^{\{0\}} = a_{n+1} \quad (6)$$

and

$$a_{n+1}^{\{k+1\}} = \frac{a_{n+1}^{\{k\}}}{a_n^{\{k\}}} \quad k \geq 0. \quad (7)$$

For example,

$$a_{n+1}^{\{1\}} = \frac{a_{n+1}}{a_n} \quad (8)$$

and

$$a_{n+1}^{\{2\}} = \frac{a_{n+1} a_{n-1}}{a_n^2} \quad n \geq 1. \quad (9)$$

It is easy to see that $a_n$ is log-convex if $a_n^{\{2\}} \geq 1$ and log-concave if $a_n^{\{2\}} \leq 1$.

At the end of their paper [1], Amdeberhan and Moll made the following conjecture:
From (7) and (10), it is easy to check that $k$ is odd and log-concave when $\{A_n\}_n\geq 1$. It follows from (5) and (8) that

In this section, we give a proof of Theorem 1.2 and present a new conjecture.

2 Proof of Theorem 1.2

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Proof. It follows from (5) and (8) that

$$A_{n+1}^{(1)} = \frac{A_{n+1}}{A_n} = \left(\prod_{j=1}^{n+1} \frac{i+j+n}{2i+j-1}\right) \left(\prod_{j=1}^{n} \frac{2i+j-1}{i+j+n-1}\right)$$

$$= \frac{3n+2}{n+1} \left(\prod_{i=1}^{n+1} \frac{1}{n+2i}\right) \left(\prod_{i=1}^{n} (i+2n+1)(i+2n)\right) \left(\prod_{j=1}^{n-1} \frac{1}{2j+n+1}\right)$$

$$= \prod_{j=1}^{n} \frac{(2n+j+1)(2n+j)}{(2j+n-1)(2j+n)} = \frac{(3n+1)!n!}{(2n+1)!(2n)!}. \quad (10)$$

From (7) and (10), it is easy to check that

$$A_{n+1}^{(2)} = \frac{3(3n+1)(3n-1)}{4(2n+1)(2n-1)},$$

$$A_{n+1}^{(3)} = \frac{(3n+1)(3n-1)(2n-3)}{(2n+1)(3n-2)(3n-4)},$$

$$A_{n+1}^{(4)} = \frac{(3n+1)(3n-1)(2n-3)(2n-1)(3n-5)(3n-7)}{(2n+1)(3n-2)^2(3n-4)^2(2n-5)},$$

$$A_{n+1}^{(5)} = \frac{(3n+1)(3n-1)(2n-3)^2(3n-5)^3(3n-7)^3(2n-7)}{(2n+1)(3n-2)^3(3n-4)^3(2n-5)^2(3n-8)(3n-10)^2},$$

$$A_{n+1}^{(6)} = \frac{(3n+1)(3n-1)(2n-3)^3(3n-5)^6(3n-7)^6(2n-7)^3(3n-11)(3n-13)}{(2n+1)(3n-2)^4(3n-4)^4(2n-5)^2(3n-8)^4(3n-10)^4(2n-3)^2(2n-9)^2}.$$

It is a routine to verify that for $1 \leq k \leq 3$,

$$A_{n+1}^{(2k)} > 1, \quad (n \geq 2k-1).$$

Conjecture 1.1 Let $A_n$ be the ASM sequence. For $0 \leq k \leq 3$, the iterated sequence $A_{n+1}^{(k)}$ is log-convex. For $k \geq 4$, the sequence $A_{n+1}^{(k)}$ is log-convex when $k$ is odd and log-concave when $k$ is even.

In this note, we show that Conjecture 1.1 is not true and we shall give a new conjecture. Our main result is the following:

Theorem 1.2 Let $\{A_n\}_{n=0}^{\infty}$ be the ASM sequence. The iterated sequence $\{A_{n+1}^{(2k)}\}_{n=0}^{\infty}$ is log-convex for $0 \leq k \leq 2$. The sequences $\{A_{n+1}^{(3)}\}_{n=3}^{\infty}$, $\{A_{n+1}^{(5)}\}_{n=5}^{\infty}$, $\{A_{n+1}^{(7)}\}_{n=7}^{\infty}$ are concave and the sequence $\{A_{n+1}^{(1)}\}_{n=1}^{\infty}$ is log-concave and convex.
Therefore, the sequences \( \{A_n\}_{n=0}^{\infty}, \{A_n^{(2)}\}_{n=2}^{\infty} \) and \( \{A_n^{(4)}\}_{n=4}^{\infty} \) are log-convex. Also, it is easy to check that for \( k = 1, 2, 3 \),

\[
A_{n+1}^{(2k+1)} + A_{n-1}^{(2k+1)} - 2A_n^{(2k+1)} < 0, \quad (n \geq 2k + 2).
\]

Hence, the sequences \( \{A_n^{(3)}\}_{n=3}^{\infty}, \{A_n^{(5)}\}_{n=5}^{\infty} \) and \( \{A_n^{(7)}\}_{n=7}^{\infty} \) are concave. At last, it is easy to verify that for \( n \geq 2 \),

\[
A_{n+1}^{(1)} + A_{n-1}^{(1)} - 2A_n^{(1)} > 0 \quad \text{and} \quad A_{n+1}^{(3)} < 1,
\]

which implies that the sequence \( \{A_n^{(1)}\}_{n=1}^{\infty} \) is log-concave and convex. This completes the proof.

To conclude this section, we give the following conjecture, which has been verified for \( n \leq 1000 \) and \( 1 \leq k \leq 10 \) by computer.

**Conjecture 2.1** Let \( \{S_n\}_{n=0}^{\infty} \) denote any sequence of \( \{A_n\}_{n=0}^{\infty}, \{PP_n\}_{n=0}^{\infty}, \{SPP_n\}_{n=0}^{\infty} \) and \( \{TSPP_n\}_{n=0}^{\infty} \). The iterated sequence \( \{S_n^{(2k)}\}_{n=2k}^{\infty} \) is log-convex and \( \{S_n^{(2k+1)}\}_{n=2k+1}^{\infty} \) is concave for any integer \( k \geq 1 \).

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**References**


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