Viscosity Approximation Methods of Common Fixed Points for an Infinite Family of Asymptotically Nonexpansive Mappings\textsuperscript{1}

Hongping Luo and Yuanheng Wang

Department of Mathematics
Zhejiang Normal University
Jinhua 321004, P.R. China
luohongping1987@163.com
yhwang@zjnu.cn

Abstract

The aim of this paper is to prove a new iterative sequence \( \{x_n\} \) under some sufficient and necessary conditions converges strongly to a common fixed point of asymptotically nonexpansive mappings \( \{T_i\}_{i=1}^{\infty} \) by using viscosity approximation methods. Our results extend and improve some recent results.

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1 Introduction

Assume that \( E \) is a real Banach space, \( E^* \) is the dual space of \( E \), \( J : E \to 2^{E^*} \) is the normalized duality mapping defined by

\[
J(x) = \{f \in E^*: \langle x, f \rangle = ||x||^2 = ||f||^2\}, x \in E.
\]

The space \( E \) is said to have a Gateaux differentiable norm, if the limit

\[
\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t}
\]

exists for each \( y \) and any \( x \) in its unit sphere \( U = \{x \in E : ||x|| = 1\}\).

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A Banach space $E$ whose norm is uniformly Gateaux differentiable, then the duality map $J$ is single-valued and norm-to-weak$^*$ uniformly continuous on bounded sets of $E$.

Recently, many authors have considered the common fixed points of a family of the nonexpansive mappings in Banach space. Some people show by using viscosity approximation methods and obtain some good results\textsuperscript{[1–4]}.

In 2008, Zhao\textsuperscript{[2]} proved the following conclusion:

Let $E$ be a uniformly smooth Banach space, $f \in \prod \mathcal{C}$, $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive mappings of $C$ into itself, such that the set $\cap_{i=1}^{N} F(T_i)$ is nonempty. Under some sufficient conditions, the iterative sequence $\{x_n\}$ defined by (1) converges strongly to a common fixed point of $T_1, T_2, \ldots, T_N$.

\begin{align*}
\begin{cases}
y_n = \beta_{n+1}x_n + (1 - \beta_{n+1})T_{n+1}x_n, & n \geq 0, \\
x_{n+1} = \alpha_{n+1}f(x_n) + (1 - \alpha_{n+1})T_{n+1}y_n, & n \geq 0.
\end{cases}
\end{align*}

The nonexpansive mapping is a special asymptotically nonexpansive mapping. Hence, if add some conditions, we can also obtain the convergence of relative sequences. In this paper, we prove, under appropriate conditions on $K, T_i, \{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$, a sequence defined by

\begin{align*}
\begin{cases}
y_n = \beta_n x_n + (1 - \beta_n)T_n x_n, & n \geq 0, \\
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, & n \geq 0.
\end{cases}
\end{align*}

strongly to $\hat{x} \in F$, which is a solution of the following variational inequality $\langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0, \forall p \in F$.

Our theorem extends\textsuperscript{[2]} to the more general class of asymptotically nonexpansive mappings.

## 2 Preliminaries

In order to prove our result, we need the following definitions and lemmas.

**Lemma 2.1**\textsuperscript{[5]} In a Banach space $E$, there holds the inequality

$$
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \ \forall x, y \in C, \text{where} \ j(x + y) \in J(x + y).
$$

**Lemma 2.2**\textsuperscript{[6]} Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence of nonnegative real numbers satisfying the property

$$
\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n \delta_n, \ n \geq 0,
$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\delta_n\}$ are such that

1. $\lim_{n \to \infty} \gamma_n = 0$, $\sum \gamma_n = \infty$;
2. $\limsup_{n \to \infty} \delta_n \leq 0$ (or $\sum |\mu_n \delta_n| < \infty$).

Then $\lim_{n \to \infty} \alpha_n = 0$. 

Lemma 2.3 [7] Let \( \{x_n\}, \{y_n\} \) be bounded sequences in a Banach space \( X \), \( \{\alpha_n\} \subset [0, 1] \) satisfying \( 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1 \), suppose

\[
(1) x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n; \quad (2) \limsup_{n \to \infty} (||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0.
\]

then \( \lim_{n \to \infty} ||y_n - x_n|| = 0 \).

Definition 1 A self-mapping \( f : C \to C \) is a contraction on \( C \), if there exists a constant \( \alpha \in (0, 1) \), such that

\[
||f(x) - f(y)|| \leq \alpha ||x - y||, \forall x, y \in C.
\]

We use \( \prod C \) to denote the collection of all contractions on \( C \).

Definition 2 [8] Let \( C \) be a nonempty convex subset of \( E \), \( T_1, T_2, \ldots, T_N \) is said to be a finite family of asymptotically nonexpansive if there exists a sequence \( \{h_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} h_n = 1 \) such that

\[
||T^n_i x - T^n_i y|| \leq h_n ||x - y||, \forall x, y \in C, i = 1, 2, \ldots, N.
\]

Denote by \( F(T_i) \) the set of fixed points of \( T_i \), which is \( F(T_i) = \{x \in C : T_i x = x\} \).

### 3 Main result

Now, we are ready to give our main results.

Theorem 3.1 Let \( K \) be a nonempty closed convex subset of a real Banach space \( E \) with a uniformly Gateaux differentiable norm Banach space. Let \( \{T_i\}_{i=1}^\infty \) be an infinite family of asymptotically nonexpansive mapping and uniform Lipschitzian from \( K \) into itself, \( F = \cap_{i=1}^\infty F(T_i) \neq \emptyset \) and \( f \in \prod K \).

Assume that the sequences \( \{\alpha_n\}, \{\beta_n\} \in (0, 1) \) satisfy the following conditions:

\( C1) \Sigma \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0; \)

\( C2) 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1; \)

\( C3) h_n - 1 = o(\alpha_n). \)

Then the sequence \( \{x_n\} \) defined by (2) converges strongly to \( \hat{x} \in F \), if and only if for any \( i \), \( \lim_{n \to \infty} ||T_i x_n - x_n|| = 0 \) holds;

And \( \hat{x} \) is a solution of the following variational inequality:

\[
\langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0, \forall p \in F.
\]
**Proof: Adequacy** The adequacy proof is divided into five steps.

Step 1 We observe \( \{x_n\} \) is bounded. Indeed, taking a fixed point \( p \) of \( F \), we have

\[
\|y_n - p\| \leq \beta_n \|x_n - p\| + (1 - \beta_n) \|T^m_i x_n - p\| \\
\leq a_n \|x_n - p\|. \\
\|x_{n+1} - p\| \leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) \|y_n - p\| \\
\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) a_n \|x_n - p\| + \alpha_n \|f(p) - p\| \\
= (a_n - (a_n - \alpha) \alpha_n) \|x_n - p\| + \alpha_n (a_n - \alpha) \|\frac{f(p) - p}{a_n - \alpha}\| \\
\leq a_n \max \{\|x_n - p\|, \|\frac{f(p) - p}{a_n - \alpha}\|\}. 
\]

Using an introduction, we have

\[
\|x_n - p\| \leq a \max \{\|x_0 - p\|, \frac{\|f(p) - p\|}{b - \alpha}\}, 
\]

where \( a_n = \beta_n + (1 - \beta_n) h_n \), \( a = \max \{a_n\} \), \( b = \min \{a_n\} = 1 \).

Hence \( \{x_n\} \) is bounded, so are the sets \( \{y_n\} \), \( \{f(x_n)\} \), and \( \{T^m_i x_n\} \).

Step 2 We claim that \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \).

Setting \( l_n = (1 - \alpha_n) \beta_n \), \( n \geq 1 \), it follows from (C1) and (C2) that

\( 0 < \lim \inf_{n \to \infty} l_n \leq \lim \sup_{n \to \infty} l_n < 1 \).

Define \( x_{n+1} = l_n x_n + (1 - l_n) z_n \), observe that

\[
z_{n+1} - z_n = \frac{x_{n+2} - l_{n+1} x_{n+1}}{1 - l_{n+1}} - \frac{x_{n+1} - l_n x_n}{1 - l_n} \\
= \left[ \alpha_{n+1} f(x_{n+1}) + (1 - \alpha_{n+1})(\beta_{n+1} x_{n+1} + (1 - \beta_{n+1}) T^m_i x_{n+1}) \right] \frac{1}{1 - l_{n+1}} \\
- \frac{l_{n+1} x_{n+1}}{1 - l_{n+1}} - \frac{\alpha_n f(x_n) + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n) T^m_i x_n) - l_n x_n}{1 - l_n} \\
= \frac{\alpha_{n+1} f(x_{n+1})}{1 - l_{n+1}} - \frac{\alpha_n f(x_n)}{1 - l_n} + \frac{(1 - \alpha_{n+1})(1 - \beta_{n+1}) T^m_i x_{n+1}}{1 - l_{n+1}} \\
- \frac{(1 - \alpha_n)(1 - \beta_n) T^m_i x_n}{1 - l_n} \\
= \frac{\alpha_{n+1} f(x_{n+1})}{1 - l_{n+1}} - \frac{\alpha_n f(x_n)}{1 - l_n} + \frac{(1 - \alpha_{n+1})(1 - \beta_{n+1}) T^m_i x_{n+1}}{1 - l_{n+1}} \\
- \frac{(1 - \alpha_n)(1 - \beta_n) T^m_i x_n}{1 - l_n} \\
= \frac{\alpha_{n+1}}{1 - l_{n+1}} (f(x_{n+1}) - T^m_i x_{n+1}) - \frac{\alpha_n}{1 - l_n} (f(x_n) - T^m_i x_n) \\
+ (T^m_i x_{n+1} - T^m_i x_n). 
\]
we have

$$
\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \| \\
\leq \frac{\alpha_{n+1}}{1 - l_{n+1}} \| f(x_{n+1}) - T_i^{n+1} x_{n+1} \| + \frac{\alpha_n}{1 - l_n} \| f(x_n) - T_i^n x_n \| \\
+ \| T_i^{n+1} x_{n+1} - T_i^{n+1} x_n \| + \| T_i^n x_n - T_i^n x_n \| - \| x_{n+1} - x_n \| \\
= \frac{\alpha_{n+1}}{1 - l_{n+1}} (\| f(x_{n+1}) \| + \| T_i^{n+1} x_{n+1} \|) + \frac{\alpha_n}{1 - l_n} (\| f(x_n) \|) \\
+ \| T_i^n x_n \|) + (h_{n+1} - 1) \| x_{n+1} - x_n \| + h_n \| T_i^n x_n - x_n \|.
$$

Using the conclusion of step 1, by (C1), $\lim_{n \to \infty} \| T_i x_n - x_n \| = 0$ and $\lim_{n \to \infty} h_n = 1$, we obtain that $\limsup_{n \to \infty} (\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \|) \leq 0$. Hence by lemma 2.3, we have $\lim_{n \to \infty} \| z_n - x_n \| = 0$.

Consequently, $\lim_{n \to \infty} \| x_{n+1} - x_n \| = \lim_{n \to \infty} (1 - l_n) \| z_n - x_n \| = 0$.

Step 3 We prove that $\lim_{n \to \infty} \| x_n - T_i^n x_n \| = 0$.

From (2), we arrive at

$$
\| y_n - x_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - y_n \| = \| x_n - x_{n+1} \| + \alpha_n \| f(x_n) - y_n \|.
$$

Since $\{f(x_n)\}$ and $\{y_n\}$ are bounded, by (C1) and $\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0$, we get $\lim_{n \to \infty} \| y_n - x_n \| = 0$.

From $y_n = \beta_n x_n + (1 - \beta_n) T_i^n x_n$, then $\lim_{n \to \infty} \| x_n - T_i^n x_n \| = 0$.

Step 4 We show that $\langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0$.

Same as [3], let $\hat{x} = \lim_{t \to 0} x_t$ with $x_t$ being the fixed point of the contraction $x = t f(x) + (1 - t) T_i^n x$, where $t \in (0, 1)$. That is $x_t = t f(x_t) + (1 - t) T_i^n x_t$.

Thanks to lemma 2.1, we have

$$
\| x_t - x_n \|^2 = \| (1 - t)(T_i^n x_t - x_n) + t (f(x_t) - x_n) \|^2 \\
\leq (1 - t)^2 \| T_i^n x_t - x_n \|^2 + 2t \langle f(x_t) - x_n, J(x_t - x_n) \rangle \\
\leq (1 - t)^2 h_n \| x_t - x_n \|^2 + g_n(t) + 2t \langle f(x_t) - x_t, J(x_t - x_n) \rangle \\
+ 2t \langle x_t - x_n, J(x_t - x_n) \rangle,
$$

where $g_n(t) = (2h_n \| x_t - x_n \| + \| T_i^n x_n - x_n \|) \| T_i^n x_n - x_n \|$.

It follows from step 2 that $\lim_{n \to \infty} g_n(t) = 0$. Then $\langle x_t - f(x_t), J(x_t - x_n) \rangle \leq \| 1 - t \|^2 h_n \| x_t - x_n \|^2 + \frac{1}{2t} g_n(t)$, we see that $\limsup_{n \to \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq \frac{1}{2} M_1$, where $M_1 \geq 0$, such that $M_1 \geq \| x_t - x_n \|^2$, $\forall t \in (0, 1), n \geq 1$.

Then

$$
\limsup_{t \to 0} \limsup_{n \to \infty} \langle x_t - f(x_t), J(x_t - x_n) \rangle \leq 0.
$$
So for any $\epsilon > 0$, $\exists \delta_1 > 0$, when $t \in (0, \delta_1)$, we get

$$\limsup_{n \to \infty} \langle x_t - f(\hat{x}), J(x_t - x_n) \rangle \leq \frac{\epsilon}{2}.$$ 

On the other hand, $x_t \to \hat{x}$ and from J is norm-to-norm uniformly continuous on bounded subsets of C, $\exists \delta_2 > 0$, such that when $t \in (0, \delta_2)$, we have

$$| \langle f(\hat{x}) - \hat{x}, J(x_n - \hat{x}) \rangle - \langle x_t - f(\hat{x}), J(x_t - x_n) \rangle |$$

$$\leq | \langle f(\hat{x}) - \hat{x}, J(x_n - \hat{x}) \rangle - \langle f(\hat{x}) - \hat{x}, J(x_n - x_t) \rangle | + | \langle f(\hat{x}) - \hat{x}, J(x_n - x_t) \rangle - \langle x_t - f(\hat{x}), J(x_t - x_n) \rangle |$$

$$\leq | \langle f(\hat{x}) - \hat{x}, J(x_n - \hat{x}) - J(x_n - x_t) \rangle | + \langle x_t - \hat{x}, J(x_n - x_t) \rangle \leq \frac{\epsilon}{2}.$$ 

Choosing $\delta = \min \{ \delta_1, \delta_2 \}, \forall t \in (0, \delta)$, we have

$$\langle u - \hat{x}, J(x_n - \hat{x}) \rangle \leq \langle x_t - u, J(x_t - x_n) \rangle + \frac{\epsilon}{2},$$

$$\limsup_{n \to \infty} \langle f(\hat{x}) - \hat{x}, J(x_n - \hat{x}) \rangle \leq \limsup_{n \to \infty} \langle x_t - f(\hat{x}), J(x_t - x_n) \rangle + \frac{\epsilon}{2} \leq \epsilon.$$

Since $\epsilon$ is chosen arbitrarily, we get $\limsup_{n \to \infty} \langle u - \hat{x}, J(x_n - \hat{x}) \rangle \leq 0$. Hence $\langle \hat{x} - f(\hat{x}), J(\hat{x} - p) \rangle \leq 0$, $\limsup_{n \to \infty} \langle f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x}) \rangle \leq 0$ holds.

**Step 5** We prove that $\lim_{n \to \infty} ||x_n - \hat{x}|| = 0$. Setting $\xi_{n+1} = \max \{ \langle f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x}) \rangle \}$, we have $\limsup_{n \to \infty} \xi_{n+1} \leq 0$.

$$||x_{n+1} - \hat{x}||^2 = ||\alpha_n (f(x_n) - \hat{x}) + (1 - \alpha_n)(\beta_n (x_n - \hat{x}) + (1 - \beta_n)(T^n x_n - \hat{x}))||^2$$

$$\leq (1 - \alpha_n)^2 [\beta_n ||x_n - \hat{x}||^2 + (1 - \beta_n)h^2 ||x_n - \hat{x}||^2 + 2\alpha_n (f(x_n) - f(\hat{x})) J(x_{n+1} - \hat{x})]$$

$$\leq (1 - \alpha_n)^2 h^2 ||x_n - \hat{x}||^2 + 2\alpha_n ||x_n - \hat{x}|| ||x_{n+1} - \hat{x}|| + \alpha_n (f(x_n) - f(\hat{x})) J(x_{n+1} - \hat{x})$$

$$\leq (1 - \alpha_n)^2 ||x_n - \hat{x}||^2 + (1 - \alpha_n)^2 (h^2 - 1) ||x_n - \hat{x}||^2 + \alpha_n (||x_n - \hat{x}||^2 + ||x_{n+1} - \hat{x}||^2) + 2\alpha_n (f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x})), $$

$$(1 - \alpha_n)||x_{n+1} - \hat{x}||^2 \leq (1 - 2\alpha_n + \alpha_n^2) ||x_n - \hat{x}||^2 + (h^2 - 1) ||x_n - \hat{x}||^2$$

$$+ \alpha_n ||x_n - \hat{x}||^2 + 2\alpha_n (f(\hat{x}) - \hat{x}, J(x_{n+1} - \hat{x})),$$

$$\leq (1 - 2\alpha_n + \alpha_n^2) ||x_n - \hat{x}||^2 + (\alpha_n^2 + h^2 - 1) ||x_n - \hat{x}||^2 + 2\alpha_n \xi_{n+1}.$$. 
which implies that
\[
\|x_{n+1} - \hat{x}\|^2 \leq (1 - \frac{2(1 - \alpha)\alpha_n}{1 - \alpha_n\alpha})\|x_n - \hat{x}\|^2 + \frac{2(1 - \alpha)\alpha_n}{1 - \alpha_n\alpha}M_2 + \frac{1}{1 - \alpha}\xi_{n+1}
\]
\[
= (1 - \gamma_n)\|x_n - \hat{x}\|^2 + \gamma_n\delta_n,
\]
where
\[
M_2 = \sup(h_n + 1)\|x_n - \hat{x}\|^2, \gamma_n = \frac{2(1 - \alpha)\alpha_n}{1 - \alpha_n\alpha},
\]
\[
\delta_n = \frac{\alpha_n + \frac{h_n - 1}{\alpha_n}}{2(1 - \alpha)}M_2 + \frac{1}{1 - \alpha}\xi_{n+1}.
\]
Since \( \{x_n\} \) is bounded, by (C1),(C3)and step 3, we have
\[
\lim_{n \to \infty} \gamma_n = 0, \sum_{n=\infty} \gamma_n = \infty; \limsup_{n \to \infty} \delta_n \leq 0.
\]
According to lemma 2.2 ,we deduce that \( \lim_{n \to \infty} \|x_n - \hat{x}\| = 0. \)

**Necessity** If \( \lim_{n \to \infty} \|x_n - \hat{x}\| = 0, \) , \( \hat{x} \in F \) and \( T_i \) is uniform Lipschitzian,
\[
\lim_{n \to \infty} \|T_ix_n - x_n\| \leq \lim_{n \to \infty} \|T_ix_n - T_i\hat{x}\| + \|T_i\hat{x} - \hat{x}\| + \|\hat{x} - x_n\|
\]
\[
\leq \lim_{n \to \infty} \|T_i\hat{x} - \hat{x}\| + \lim_{n \to \infty} (L + 1)\|\hat{x} - x_n\| \leq 0.
\]

Hence the proof of Theorem 3.1 is completed. \( \square \)

**Remark 1** Our paper improve[1 – 4], such as extending a finite family of nonexpansive mappings to an infinite family of asymptotically nonexpansive mappings.

**References**


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