Multi-Period Mean-Variance Portfolio Selection
with a Benchmark Process

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Abstract

Mean-variance portfolio selection with a benchmark process is to maximize the expected final surplus subject to a given variance or, minimize the variance of final surplus subject to a given expectation. We introduce the benchmark process and define the surplus process as the difference between a wealth process and the benchmark process. We derive the optimal portfolio and the mean-variance efficient frontier in closed form via investigating the discrete-time version of the stochastic linear-quadratic control theory.

Keywords: Contingent claim; optimal hedging strategy; mean-variance criterion; efficient frontier

1 Introduction

The mean-variance criterion is proposed by Markowitz [1] for a single period, and is extended to the multi-period case, see Yin and Zhou[2], Li and Ng[3], Leippols et al.[4]. However, a serious difficulty in the multi-period case is the variance’s non-separability, see chen [5] for reference. Li and Ng[3] introduces the embedding technique to overcome this difficulty. Li’s embedding technique has many far-reaching consequences. For instance, Costa and Nabholz[6]consider the mean-variance optimization with inter-temporal restrictions. Costa and Araujo[7] employ the embedding technique to solve the portfolio selection problem with markov switching parameters. Zhu et al.[8] propose a dynamic multi-period mean-variance portfolio selection with bankruptcy. However, The hedging strategy selection of contingent claim is not taken into account in above literatures which only concentrate on the portfolio selection. The benchmark process is an important factor in most of the real-world situations, thus the introduction of the benchmark process in a selection problem will make the mean-variance portfolio selection problem more practical. Moreover, Liability which is a special case of the benchmark process is introduced and is abound in the literature. Leippold et al.[5] study the mean-variance portfolio selection with assets and liability in a discrete-time setting. The main goal of this paper is to generalize the wealth process to a surplus process and maximize the expected final surplus for a given variance level. The surplus process is defined as the difference between the wealth process and the benchmark process which can be considered as liability in an asset-liability problem. The main difficulty of above mentioned problem comes from the benchmark process’s driving factors.
Due to introducing the benchmark process, the surplus process described in this paper is the function of the wealth process and the benchmark process, while in a asset-liability problem, the surplus process is only related to the wealth process. By generalizing the stochastic control theory in [9], we formulate the problem as a general stochastic LQ control question whose solution is closely related to the corresponding Riccati-typed equations, and derive the mean-variance efficient frontier in closed form.

2 Model

Consider a financial model with $N$ assets on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. An investor is only interested in $m + 1$ assets. Of $m + 1$ assets, the asset labeled by $i = 0$ is the risk-free asset and the remaining assets, labeled by $i = 1, 2, \cdots, m$, are risky assets. The risk-free asset $S_0(t)$ satisfies the following differential equation:

$$S_0(t + 1) = S_0(t)(1 + \alpha_0(t)), \quad S_0(0) = s_0. \tag{1}$$

The price processes $S_i(t), i = 1, 2, \cdots, m$ satisfy the following stochastic differential equations:

$$S_i(t + 1) = S_i(t) + S_i(t)(\alpha_i(t) + \sum_{j=1}^{n} \sigma_i^j Z_j(t)), \quad S_i(0) = s_i, \tag{2}$$

where $Z(t) = (Z_1(t), Z_2(t), \cdots, Z_n(t))^\prime$ is an $n$-dimensional independent random vector with zero mean and covariance $I$ (identity matrix), and the superscript $\prime$ represents the transpose of a vector or a matrix.

Consider that an investor with an initial wealth $x_0$ invests in the financial market. The investor is allowed to adjust portfolio at dates $0, 1, 2, \cdots, T - 1$, and short-selling is allowable. Define $\pi(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_m(t))^\prime$, and let $\pi_i(t), i = 1, 2, \cdots, m$, be the amount invested in the $i$th risky asset at the beginning of the $t$th time period. For a self-financing admissible control portfolio $\pi(t) \in \mathcal{A}(t)$, the wealth process $x(t)$ is represented by

$$x(t + 1) = x(t)(1 + \alpha_0(t)) + \tilde{\alpha}(t)\pi(t) + \pi(t)^\prime \sigma(t)Z(t), \tag{3}$$

where $\tilde{\alpha}(t) = (\alpha_1(t) - \alpha_0(t), \alpha_2(t) - \alpha_0(t), \ldots, \alpha_m(t) - \alpha_0(t))$ and $\sigma(t) = (\sigma_i^j(t))_{m \times n}$.

The investor selects the benchmark process $l(t)$ with an initial value $l_0$. The benchmark process follows,

$$l(t + 1) = l(t)(1 + \beta(t)) + l(t)\sigma_L(t)^\prime Z(t), \quad l(0) = l_0, \tag{4}$$

where $\sigma_L(t) = (d^1(t), d^2(t), \ldots, d^m(t))^\prime$.

The surplus process is defined as

$$Y(t) = x(t) - l(t). \tag{5}$$

By subtracting (3) from (4), the stochastic differential equation for the surplus can be derived as

$$Y(t + 1) = (1 + \alpha_0(t))Y(t) + (\alpha_0(t) - \beta(t))l(t) + \tilde{\alpha}(t)\pi(t) + (-\sigma_L(t)^\prime l(t) + \pi(t)^\prime \sigma(t))Z(t),$$

$$Y(0) = x_0 - l_0. \tag{6}$$
Hence, the mean-variance portfolio selection with the benchmark process can be expressed as

\[ P(\zeta) : \min \mathbb{E} \left[ Y(T)^2 - \lambda Y(T) \right] \]

subject to \( \mathbb{E} Y(T) \geq \zeta, \pi(t) \in \mathcal{A}_t \) and (6).

Following Li and Ng [3], the problems \( P(\zeta) \) is equivalent to the following problem

\[ P(\omega) : \min \mathbb{E} \left[ -Y(T) + \omega \mathbb{E} Y(T) \right] \]

subject to \( \mathbb{E} Y(T) \geq \zeta, \pi(t) \in \mathcal{A}_t \) and (6).

for some strictly positive parameter \( \omega \). Moreover, if \( \pi^* \) solves \( P(\omega) \), it solves \( P(\zeta) \) with \( \zeta = \mathbb{E} Y(T) \mid \pi^* \).

3 Solution of an Auxiliary Problem

The stochastic problem \( P(\omega) \) cannot be solved directly by dynamic programming, since it is nonseparable and contains the term \( \text{Var}(Y_T) \). As pointed out in [3], a solution scheme is to embed \( P(\omega) \) into an auxiliary problem, investigate the relationship between the solution \( P(\omega) \) and the auxiliary problem and search for the solution of original problem. We adopt the same procedure in this paper, and consider the following auxiliary problem

\[ P(\lambda, \omega) : \min \mathbb{E} \left[ \omega Y(T)^2 - \lambda Y(T) \right] \]

subject to \( \mathbb{E} Y(T) \geq \zeta, \pi(t) \in \mathcal{A}_t \) and (6),

where \( \lambda \in \mathbb{R}, \omega > 0 \) are given beforehand. The optimal value function of the problem \( P(\lambda, \omega) \) is defined as

\[ V(t, y, l) = \min_{\pi(t) \in \mathcal{A}_t} \mathbb{E} \left[ \omega Y(T)^2 - \lambda Y(T) \mid \mathcal{F}_t \right]. \] (7)

**Theorem 3.1** The optimal control strategy of \( P(\lambda, \omega) \) is given by

\[
\begin{align*}
\pi^*(t) &= -(\hat{a}(t) - \lambda \hat{a}(t)) - \sigma(t) \hat{a}(t) - (1 + \alpha(t)) \hat{a}(t) + \frac{\lambda}{\omega} \prod_{k=1}^{T-1} (1 + \alpha(k)) \hat{a}(t) \\
&\quad + [(1 + \alpha(t)) \hat{a}(t) - \prod_{k=1}^{T-1} \frac{(1 + \beta(k)) (1 - A(k)) - B(k)}{(1 + \alpha(k)) (1 - A(k))} ((1 + \beta(t)) \hat{a}(t) + \sigma(t) \hat{a}(t) + \mathcal{F}_t))]l(t) \}
\end{align*}
\] (8)

where \( A(t) = \hat{a}(t) (\hat{a}(t)' (\hat{a}(t) + \sigma(t) \hat{a}(t)')^{-1} \hat{a}(t), B(t) = \hat{a}(t) (\hat{a}(t)' (\hat{a}(t) + \sigma(t) \hat{a}(t)')^{-1} \hat{a}(t) \hat{a}(t)^{-1} \sigma(t) \hat{a}(t)) \), \( C(t) = (\sigma(t) \hat{a}(t) (\hat{a}(t)' (\hat{a}(t) + \sigma(t) \hat{a}(t)')^{-1} \hat{a}(t) \hat{a}(t)^{-1} \sigma(t) \hat{a}(t)) \).

**Proof** Firstly, show that the value function is given by

\[ V(t, y, l) = \frac{1}{2} P(t) y(t)^2 + Q(t) y(t) + K(t) y(t) l(t) + \frac{1}{2} G(t) l(t)^2 + g(t) l(t) + R(t), \] (9)
where

\[ P(t) = P(t + 1)(1 + \alpha_0(t))^2(1 - A(t)), \]
\[ P(T) = 2\omega, \]
\[ Q(t) = Q(t + 1)(1 + \alpha_0(t))(1 - A(t)), \]
\[ Q(T) = -\lambda, \]
\[ K(t) = P(t + 1)[(1 + \alpha_0(t))(\alpha_0(t) - \beta(t))(1 - A(t)) + (1 + \alpha_0(t))B(t)] + K(t + 1)[(1 + \alpha_0(t))(1 + \beta(t))(1 - A(t)) - (1 + \alpha_0(t))B(t)], \]
\[ K(T) = 0, \]
\[ G(t) = P(t + 1)[(\alpha_0(t) - \beta(t))^2 + \sigma_L(t)\sigma_L(t) - (\alpha_0(t) - \beta(t))^2A(t) + 2(\alpha_0(t) - \beta(t))B(t) - C(t)] + 2K(t + 1)[(\alpha_0(t) - \beta(t))(1 + \beta(t))(1 - A(t)) - \sigma_L(t)\sigma_L(t) - (\alpha_0(t) - 2\beta(t) - 1)B(t) - C(t)] - \frac{K(t + 1)^2}{2T(t + 1)}[(1 + \beta(t))^2A(t) + 2(1 + \beta(t))B(t) + C(t)] + \frac{1}{2}G(t + 1)((1 + \beta(t))^2 + \sigma_L(t)\sigma_L(t)), \]
\[ G(T) = 0, \]
\[ R(t) = R(t + 1) - \frac{1}{2}\frac{Q(t + 1)^2}{P(t + 1)}A(t), \]
\[ R(T) = 0. \] (10)

The above result can be proved by mathematical induction on \( t \). For \( t = T \), we have that

\[ V(T, y, l) = \omega y(T)^2 - \lambda y(T) \]
\[ = \frac{1}{2}P(T)y(T)^2 + Q(T)y(T) + K(T)y(T)l(T) + \frac{1}{2}G(T)y(T)^2 + g(T)y(T) + R(T) \]

proving (9) for \( t = T \). Suppose that (9) holds for \( t + 1 \); then

\[ V(t, y, l) = \min_{\pi(t) \in \mathcal{A}_t} \mathbb{E}[\omega Y(T)^2 - \lambda Y(T)|\mathcal{F}_t] \]
\[ = \min_{\pi(t) \in \mathcal{A}_t} \mathbb{E} \left[ \min_{\pi(t+1) \in \mathcal{A}_{t+1}} \mathbb{E}[\omega Y(T)^2 - \lambda Y(T)|\mathcal{F}_{t+1}] | \mathcal{F}_t \right] \]
\[ = \min_{\pi(t) \in \mathcal{A}_t} \mathbb{E} \left[ V(t + 1, y(t + 1), l(t + 1)) | \mathcal{F}_t \right] \]
\[ = \min_{\pi(t) \in \mathcal{A}_t} \mathbb{E} \left[ \frac{1}{2}P(t + 1) y(t + 1)^2 + Q(t + 1) y(t + 1) + K(t + 1) y(t + 1) l(t + 1) \right. \]
\[ \left. + \frac{1}{2}G(t + 1) y(t + 1)^2 + g(t + 1) y(t + 1) l(t + 1) + R(t + 1) | \mathcal{F}_t \right] \]

Notice that

\[ \mathbb{E} \left[ \frac{1}{2}P(t + 1) y(t + 1)^2 + Q(t + 1) y(t + 1) + K(t + 1) y(t + 1) l(t + 1) \right. \]
\[ \left. + \frac{1}{2}G(t + 1) y(t + 1)^2 + g(t + 1) y(t + 1) l(t + 1) + R(t + 1) | \mathcal{F}_t \right] \]

and

\[ \mathbb{E}[Z(t)^2|\mathcal{F}_t] = 1, \mathbb{E}[Z(t)|\mathcal{F}_t] = 0, \]

thus

\[ V(t, y, l) = \frac{1}{2}P(t + 1)(1 + \alpha_0(t))^2 y(t)^2 + \frac{1}{2}P(t + 1)((\alpha_0(t) - \beta(t))^2 + \sigma_L(t)\sigma_L(t)) \]
\[ + K(t + 1)((\alpha_0(t) - \beta(t))(1 + \beta(t)) - \sigma_L(t)\sigma_L(t)) + \frac{1}{2}G(t + 1)((1 + \beta(t))^2 + \sigma_L(t)\sigma_L(t)) y(t)^2 \]
\[ + [P(t + 1)(1 + \alpha_0(t))(\alpha_0(t) - \beta(t)) + K(t + 1)(1 + \alpha_0(t))(1 + \beta(t))] y(t) l(t) \]
\[ + Q(t + 1)(1 + \alpha_0(t)) y(t) + [Q(t + 1)(\alpha_0(t) - \beta(t)) + g(t + 1)(1 + \beta(t))] y(t) l(t) + R(t + 1) \]
\[ + \min_{\pi(t) \in \mathcal{A}_t} f(\pi(t)), \]
where the function \( f(\pi(t)) \) is given by
\[
f(\pi(t)) = \frac{1}{2} P(t+1) \pi(t)'(\tilde{\alpha}(t)'\tilde{\alpha}(t) + \sigma(t)\sigma(t)')\pi + \{P(t+1)(1 + \alpha_0(t))\tilde{\alpha}(t)y(t) \\
+ [P(t+1)(\alpha_0(t) - \beta(t))\tilde{\alpha}(t) - P(t+1)(\sigma(t)\sigma_L(t))'] \\
+ K(t+1)(1 + \beta(t))\tilde{\alpha}(t) + K(t+1)(\sigma(t)\sigma_L(t))']l(t) + Q(t+1)\tilde{\alpha}(t)\} \pi(t).
\]

Substituting (12) into (11) yields
\[
\pi^*(t) = -(\tilde{\alpha}(t)'\tilde{\alpha}(t) + \sigma(t)\sigma(t)')^{-1}\{(1 + \alpha_0(t))\tilde{\alpha}'(t)y(t) + \frac{Q(t+1)}{P(t+1)}\tilde{\alpha}'(t) \\
+ [\{(\alpha_0(t) - \beta(t) + \frac{K(t+1)}{P(t+1)}(1 + \beta(t))\tilde{\alpha}(t) + (\frac{K(t+1)}{P(t+1)} - 1)(\sigma(t)\sigma_L(t))\}l(t)]\}.
\]

Thus,
\[
V(t, y, l) = \frac{1}{2} P(t+1)\{(1 + \alpha_0(t))^2(1 - A(t))\}y(t)^2 + \frac{Q(t+1)}{P(t+1)}\{(1 + \alpha_0(t))\{1 - A(t)\}y(t) \\
+ (P(t+1)\{(1 + \alpha_0(t))(\alpha_0(t) - \beta(t))\{1 - A(t)\} + (1 + \alpha_0(t))B(t) \\
+ \{P(t+1)\{(1 + \alpha_0(t))(1 + \beta(t))\{1 - A(t)\} - (1 + \alpha_0(t))B(t)\}y(t)l(t) \\
+ \frac{1}{2}\{P(t+1)\{(1 + \alpha_0(t) - \beta(t))^2 + \sigma_L(t)\sigma_L(t) - (\alpha_0(t) - \beta(t))^2\}A(t) + 2(\alpha_0(t) - \beta(t))B(t) - C(t) \\
+ 2K(t+1)\{(\alpha_0(t) - \beta(t))(1 + \beta(t))(1 - A(t)) - \sigma_L(t)\sigma_L(t) - (\alpha_0(t) - 2\beta(t) - 1)B(t) - C(t) \\
- \frac{K(t+1)}{P(t+1)}\{(1 + \beta(t))^2A(t) + 2(1 + \beta(t))B(t) + C(t)\} + \frac{1}{2}Q(t+1)(1 + \beta(t))^2 + \sigma_L(t)\sigma_L(t))\}l(t)^2 \\
+ \{Q(t+1)\{(\alpha_0(t) - \beta(t))(1 - A(t)) - B(t)\} + \frac{Q(t+1)K(t+1)}{P(t+1)}[1 + \beta(t)]A(t) - B(t)] \\
+ g(t+1)(1 + \beta(t))\}l(t) + R(t+1) - \frac{1}{2} \frac{Q(t+1)^2}{P(t+1)} A(t) \\
= \frac{1}{2} P(t)y(t)^2 + Q(t)y(t) + K(t)y(t)l(t) + \frac{1}{2} G(t)l(t)^2 + g(t)l(t) + R(t)
\]

proving the desired result.

Solving (10) leads to
\[
P(t+1) = 2\omega \prod_{k=t+1}^{T-1} [(1 + \alpha_0(k))^2(1 - A(k))],
\]
\[
Q(t+1) = -\lambda \prod_{k=t+1}^{T-1} [(1 + \alpha_0(k))(1 - A(k))],
\]
\[
K(t+1) = -P(t+1) = -2\omega \prod_{k=t+1}^{T-1} [(1 + \alpha_0(k))(1 + \beta(k))(1 - A(k)) - B(k)].
\]

Thus
\[
\frac{Q(t+1)}{P(t+1)} = -\frac{1}{2} \frac{Q(t+1)K(t+1)}{P(t+1)} \prod_{k=t+1}^{T-1} \frac{1}{1 + \alpha_0(k)},
\]
\[
\frac{K(t+1) - P(t+1)}{P(t+1)} = -\frac{1}{2} \frac{Q(t+1)^2}{P(t+1)(1 + \alpha_0(k))(1 - A(k))}.\]

This completes the proof.
4 Solution of Original Problem

In this section, we present the solution of problem $P(\omega)$ and derive the optimal strategy and the efficient frontier of original problem. From [3], the relationship between $P(\omega)$ and $P(\lambda, \omega)$ can be shown as follow.

**Proposition 4.1** Suppose that $\pi^*$ is the optimal solution of $P(\omega)$. Then, $\pi^*$ is the optimal solution of $P(\lambda, \omega)$, with

$$\lambda^* = 1 + 2\omega E(Y^*(T)).$$

Moreover, if $\pi^*$ is the optimal solution of $P(\lambda, \omega)$, a necessary condition for $\pi^* \in P(\omega)$ is that (13) holds.

In order to obtain $\lambda^*$ such that (13) holds, we evaluate $E(Y^*(T))$. Substituting (8) into (6) and taking expectation, we obtain that

$$EY(T) = (a - b)x_0 + \frac{\lambda}{2\omega}c,$$

where $a = \prod_{k=0}^{T-1} (1 + \alpha_0(k))(1 - A(k))$, $b = \prod_{k=0}^{T-1} [(1 + \beta(k))(1 - A(k)) - B(k)], c = 1 - \prod_{k=0}^{T-1} (1 - A(k))$.

From proposition 4.1, we have that

$$\lambda^* = \frac{1 + 2\omega(ax_0 - bl_0)}{1 - c}.$$

Then, the theorem as below can be derived.

**Theorem 4.2** The optimal control strategy of $P(\omega)$ is given by

$$\pi^*(t) = -(\tilde{\alpha}(t)\tilde{\alpha}(t) + \sigma(t)\sigma(t))^{-1}\{1 + \alpha_0(t)\tilde{\alpha}'(t)g(t) - \frac{1}{1-c}(ax_0 - bl_0 + \frac{\lambda}{2\omega}) \prod_{k=t+1}^{T-1} (1 + \alpha_0(k))\tilde{\alpha}'(t)

+[(1 + \alpha_0(t))\tilde{\alpha}(t) - \prod_{k=t+1}^{T-1} \frac{(1 + \beta(k))(1 - A(k)) - B(k)}{(1 + \alpha_0(k))(1 - A(k))}((1 + \beta(t))\tilde{\alpha}'(t) + (\sigma(t)\sigma(t)))]l(t)\}.$$

Similarity, we have

$$\text{Var}Y(T) = \frac{c}{1 - c}(a - b)^2x_0^2 + (e - ef - \frac{1}{1 - c}b^2)x_0^2 + \left(\frac{\lambda}{2\omega}\right)^2(c - c^2) - \frac{2\lambda}{2\omega}c(a - bx_0)x_0. \quad (15)$$

where

$$e = \prod_{k=0}^{T-1} [(1 + \beta(k))^2 + \sigma_L(k)^2\sigma_L(k)],$$

$$f = \sum_{k=0}^{T-1} \frac{[(1 + \beta(T-k))(1 - A(T-k) - B(T-k))^2]}{(1 - A(T-k))(1 + \beta(T-k))^2 + \sigma_L(T-k)^2\sigma_L(T-k))} \prod_{k=0}^{T-1} \frac{[(1 + \beta(T_k))(1 - A(T-k) - B(T-k))^2]}{(1 + \beta(T-k))^2 + \sigma_L(T-k)^2\sigma_L(T-k))}.$$
Theorem 4.3 The optimal control strategy of $P(\zeta)$ is given by

$$
\pi^*(t) = -(\hat{\alpha}(t)'\hat{\alpha}(t) + \sigma(t)\sigma(t)')^{-1}\{(1 + \alpha_0(t))\hat{\alpha}'(t)\gamma(t) - \frac{1}{2\omega^2}(ax_0 - bl_0 + \frac{1}{2\omega}) \prod_{k=t+1}^{T-1}(1 + \alpha_0(k))\hat{\alpha}'(t) \\
+ [(1 + \alpha_0(t))\hat{\alpha}'(t) - \prod_{k=t+1}^{T-1} \frac{1 + \beta(k) - B(k)}{(1 + \alpha_0(k)(1 - A(k)))}\{(1 + \beta(t))\hat{\alpha}'(t) + (\sigma(t)\sigma_L(t))\}]\},$$

where

$$
\omega^* = \frac{c}{2((1 - c)\zeta - (ax_0 - bl_0))}.
$$

Moreover, the mean-variance efficient frontiers for $P(\zeta), P(\omega)$ and $P(\lambda, \omega)$ are all expressed by

$$
\text{Var}Y(T) = \frac{1}{c} (EY(T) - \frac{ax_0 - bl_0}{c})^2 + (e - ef - \frac{1}{1 - c} b^2)t_0^2.
$$

Proof From theorem 4.2, $\gamma^* = \frac{1 + 2\omega(ax_0 - bl_0)}{2\omega(1 - c)}$. Thus,

$$
\text{Var}Y(T) = \frac{c}{4\omega^2(1 - c)} + (e - ef - \frac{1}{1 - c} b^2)t_0^2,
$$

$$
EY(T) = \frac{ax_0 - bl_0}{1 - c} + \frac{c}{2\omega(1 - c)}.
$$

Given $\delta$ or $\zeta$, the associated $\omega$ can be calculated using (18) or (19). The mean-variance efficient frontier given by (17) can be obtained by eliminating the parameter $\omega$ in (18) and (19).

References


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