Common Extension of a Finite Collection of
Uniformly Continuous Functions with Pair Wise
Disjoint Domains

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Abstract. In [1], it was proved that any uniformly continuous function defined on a subset of a metric space \(X\) with values in \(l_\infty(\Omega)\), with modulus of continuity dominated by a nondecreasing subadditive function \(w(t)\) satisfying \(\lim_{t \to 0} w(t) = 0\), can be extended to a uniformly continuous function \(F\) on the whole space whose modulus of continuity \(w_F(t) \leq w(t)\). In this paper we give a general way to construct a common uniformly continuous extension for a finite collection of uniformly continuous functions with pair wise disjoint domains and with modulus of continuity dominated by a certain subadditive function.

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Notations and basic definitions

1. By \(l_\infty(\Omega)\) it is denoted the normed space of all bounded real valued functions with domain \(\Omega\) equipped with the norm \(\|f\| = \sup_{x \in \Omega}|f(x)|\) \(\forall f \in l_\infty(\Omega)\).

Definition (1): Modulus of continuity [1].
For a map $f$ from a metric space $(X,d)$ into a metric space $(Y,\rho)$, the modulus of continuity is the function $w_f(t) = \sup\{\rho(f(x), f(y)) : d(x,y) \leq t\}$. The map $f$ is uniformly continuous if there is a $t_0 > 0$ such that $w_f(t) < \infty$ for $t < t_0$, and if $\lim_{t \to 0+} w_f(t) = 0$.

**Some properties of Modulus of Continuity** (see [2], [3], [4], [5], [6])

For any real valued functions $f, g$, defined on a metric space $X$, it is true that,

1. $w_{cf}(t) = |c|w_f(t)$, $c$ is a constant.
2. $w_{f+g}(t) \leq w_f(t) + w_g(t)$.
3. $w_{fg}(t) \leq \|g\|w_f(t) + \|f\|w_g(t)$, provided that $f$ and $g$ are bounded on $X$.

**Theorem 1** [1]: (i) Every metric space $X$ is isometric to a subset of $l_\infty(\Omega)$ for some set $\Omega$.

(ii) Let $Z$ be a subset of the metric space $Y$, and let $w$ be a nondecreasing subadditive function satisfying $\lim_{t \to 0} w(t) = 0$. If $f : Z \to l_\infty(\Omega)$ is a uniformly continuous function with modulus of continuity $w_f(t) \leq w(t)$, then $f$ can be extended to a uniformly continuous function $F : Y \to l_\infty(\Omega)$ whose modulus of continuity $w_F(t) \leq w(t)$.

**Technical lemmas**

To prove our main results we need to the following lemma.

**Lemma(1).** For any real-valued functions $f, g$ defined on a metric space $X$ it is true that $w_{\max(f,g)}(t) \leq \max\{w_f(t), w_g(t)\}$.

**Proof:** Let us decompose $X$ into the following two sets:

$X_1 = \{x : f(x) \geq g(x)\}$, $X_2 = \{x : f(x) \leq g(x)\}$.

For the function $\varphi(x,y) = \{\max\{f(x), g(x)\} - \max\{f(y), g(y)\}\}$ we will consider the following four cases:

Case (1): $x, y \in X_1$, this means that $f(x) \geq g(x)$ and $f(y) \geq g(y)$ and hence $\varphi(x,y) = |f(x) - f(y)|$

Case (2): $x, y \in X_2$ this means that $g(x) \geq f(x)$ and $g(y) \geq f(y)$. In this case $\varphi(x,y) = |g(x) - g(y)|$

Case (3): $x \in X_1, y \in X_2$. In this case $f(x) \geq g(x)$ and $g(y) \geq f(y)$. So $\varphi(x,y) = |f(x) - g(y)|$

Case (4): $x \in X_2, y \in X_1$ and so $g(x) \geq f(x) and f(y) \geq g(y)$ then $\varphi(x,y) = |g(x) - f(y)|$

For case (3) we get that $|f(x) - g(y)| = \begin{cases} f(x) - g(y) ; & \text{for } f(x) \geq g(y) \\ g(y) - f(x) ; & \text{for } g(y) \geq f(x) \end{cases}$
But since, \( g(y) - f(x) \leq g(y) - g(x) \leq |g(y) - g(x)| \), and
\[ f(x) - g(y) \leq |f(x) - f(y)| \] (2)
Then from (1), (2) we get
\[ |f(x) - g(y)| \leq \max\{|f(y) - f(x)|, |g(y) - g(x)|\} \] (3)
Similarly for case four we get that
\[ |g(x) - f(y)| \leq \max\{|f(y) - f(x)|, |g(y) - g(x)|\} \] (4)
So in all cases we get that \( \varphi(x, y) \leq \max\{|f(y) - f(x)|, |g(y) - g(x)|\} \)
From this relation we get the following
\[ w_{\max(f,g)}(t) = \sup\{\max\{|f(x), g(x)| - \max\{f(y), g(y)|\}; d(x, y) \leq t\} = \sup\{\varphi(x, y); d(x, y) \leq t\} \leq \sup\{\max\{|f(y) - f(x)|, |g(y) - g(x)|\}; d(x, y) \leq t\} = \max\{\sup\{|f(y) - f(x)|; d(x, y) \leq t\}, \sup\{|g(y) - g(x)|; d(x, y) \leq t\}\} = \max\{w_f(t), w_g(t)\} \].

**Lemma (2).** If \( X \) is a bounded metric space then for each point \( x_0 \in X \), we can choose a bounded real valued function \( h \in l_\infty(X) \) such that \( h(x_0) = 0 \), \( \|h\| = 1 \), and \( 0 \leq h(x) \leq 1 \), for every \( x \in X \).

**Proof.** From (i) of Theorem(1) we have the isometry \( j: X \to l_\infty(X) \) which is defined as \( j(x) = j_x \in l_\infty(X) \) such that \( j_x(y) = d(x, y) - d(y, x_0) \forall y \in X \), \( \|j_x\| = d(x, x_0) \), let \( 0 < \alpha = \sup_{x \in X} d(x, x_0) < \infty \). Now we have the following function \( h: X \to \mathbb{R} \) is defined by \( h(x) = \left(\frac{1}{\alpha}\right) \|j_x\| = \left(\frac{1}{\alpha}\right) d(x, x_0) \) from this definition we get that :
\[ h(x_0) = 0, \forall x \in X, h(x) = |h(x)| = \left(\frac{1}{\alpha}\right) d(x, x_0) \leq \left(\frac{1}{\alpha}\right) \alpha = 1 \] and
\[ \|h\| = \sup_{x \in X} |h(x)| = \sup_{x \in X} \left(\frac{1}{\alpha}\right) d(x, x_0) = \left(\frac{1}{\alpha}\right) \sup_{x \in X} d(x, x_0) = 1. \]

**Remark (1):** Through the proof of (ii) in Theorem1 it was shown that the desired extension \( F: Y \to l_\infty(\Omega) \) may be taken as \( F(y) = \inf \{ f(z) + w(d(z, y)); z \in Z \} \). In fact one can simply show that another way of giving the desired extension is \( F(y) = \sup \{ f(z) + w(d(z, y)); z \in Z \} \).

**Main results**

**Theorem(2):** Let \( Z \) be a subset of the compact metric space \( Y \), and let \( w \) be a nondecreasing subadditive function satisfying \( \lim_{t \to 0} w(t) = 0 \). If \( f: Z \to l_\infty(\Omega) \) is uniformly continuous and that \( w_f(t) = w(t) \), Then \( f \) can be extended to a uniformly continuous function \( F: Y \to l_\infty(\Omega) \) whose modulus of continuity \( w_f(t) = w(t) \) and \( \|F\| = \|f\| \).
\textbf{Proof.} By considering each coordinate separately, it suffices to prove this theorem for real-valued functions; taking \( \mathbb{R} \) as the target space instead of \( L_{\infty}(\Omega) \). From the previous remark we consider the two uniformly continuous extensions for \( f \)
\[ F_1(y) = \inf \{ f(z) + w(d(z,y)) : z \in Z \}, \]
\[ F_2(y) = \sup \{ f(z) - w(d(z,y)) : z \in Z \}, \]
which are bounded on the compact space \( Y \) and satisfying \( w_{F_1}(t) \leq w(t) \), \( w_{F_2}(t) \leq w(t) \). Taking \( F = \frac{1}{2} (F_1 + F_2) \), one gets a uniformly continuous extension of \( f \) for which according to the properties of modulus of continuity satisfies
\[ w_F(t) = w_{\frac{1}{2}(F_1+F_2)}(t) \leq \frac{1}{2} \{ w_{F_1}(t) + w_{F_2}(t) \} \leq w(t). \]
Moreover, this extension \( F \) satisfies \( \|F\| = \|f\| \). In fact, From the definitions of \( F_1 \) and \( F_2 \), \( \forall \epsilon > 0 \exists \ z_1, z_2 \in Z : F(y) < f(z_1) + \epsilon \) and \(-F(y) < f(z_2) + \epsilon \) From these relations for any \( y \in Y \) we get,
\[ |F(y)| = \max\{F(y), -F(y)\} \leq \max\{f(z_1) + \epsilon, -f(z_2) + \epsilon\} \]
\[ \leq \max\{f(z_1), -f(z_2)\} + \epsilon \leq \max\{|f(z_1)|, |f(z_2)|\} + \epsilon \]
\[ \leq \sup_{z \in Z} |f(z)| + \epsilon = \|f\| + \epsilon \]
The arbitrariness of \( \epsilon \) completes the proof. \( \square \)

\textbf{Theorem (3).} Let \( Z_1, Z_2 \) be two disjoint closed subsets of a compact metric space \( X \), and let \( f_1:Z_1 \to L_{\infty}(X), f_2:Z_2 \to L_{\infty}(X) \) be two uniformly continuous functions satisfying \( w_{f_i}(t) \leq w(t), i = 1, 2 \) where \( w \) is a nondecreasing subadditive function satisfying \( \lim_{t \to 0} w(t) = 0 \). Then there exists a uniformly continuous extension \( F:X \to L_{\infty}(X) \) for \( f_1, f_2 \) with modulus of continuity satisfying
\[ w_F(t) \leq w(t) + \max\{\|f_1\|, \|f_2\| \}. \]

\textbf{Proof.} By considering each coordinate separately, it suffices to prove this theorem for \( \mathbb{R} \) as the target space instead of \( L_{\infty}(X) \). Let \( h_1:X \to [0,1], h_2:X \to [0,1] \) be defined as
\[ h_1(x) = \frac{d(Z_2,x)}{d(Z_2,x)+d(Z_1,x)}, h_2(x) = \frac{d(Z_1,x)}{d(Z_1,x)+d(Z_2,x)}, \]
where \( d(Z_\nu,x) = \inf\{d(z,x) : z \in Z_\nu\}, \nu = 1, 2 \) are the uniformly continuous distances from \( x \) to \( Z_\nu, \nu = 1, 2 \). From the definition we see that \( \|h_1\| = \|h_2\| = 1 \)
and
\[ w_{h_1}(t) = \sup\{ |h_1(x) - h_1(y)| : d(x,y) \leq t \} = \]
\[ = \sup\left\{ \left| \frac{d(Z_2,x)}{d(Z_2,x)+d(Z_1,x)} - \frac{d(Z_2,y)}{d(Z_2,y)+d(Z_1,y)} \right| : d(x,y) \leq t \right\} \]
\[ = \sup\left\{ \left| \frac{d(Z_2,x)d(Z_1,y) - d(Z_2,y)d(Z_1,x)}{(d(Z_2,x)+d(Z_1,x))(d(Z_2,y)+d(Z_1,y))} \right| : d(x,y) \leq t \right\} \leq 1 \]
Similarly, \( w_{h_2}(t) \leq 1 \). According to theorem (2) we get two extensions \( F_1:X \to L_{\infty}(X), F_2:X \to L_{\infty}(X) \) for which \( \|f_i\| = \|F_i\|, i = 1, 2 \). Taking \( F = \max\{h_1F_1, h_2F_2\} \) we see that it is uniformly continuous extension of \( f_1 \) and \( f_2 \) with
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\[ w_F(t) = w_{\text{max}(h_1 F_1, h_2 F_2)}(t) \leq \text{max}\{w_{h_1 F_1}(t), w_{h_2 F_2}(t)\} \]

\[ \leq \text{max}\{\|h_1\|w_{F_1}(t) + \|F_1\|w_{h_1}(t), \|h_2\|w_{F_2}(t) + \|F_2\|w_{h_2}(t)\} \]

\[ = \text{max}\{w_{F_1}(t) + \|F_1\|w_{h_1}(t), w_{F_2}(t) + \|F_2\|w_{h_2}(t)\} \]

\[ \leq w(t) + \text{max}\{|\|F_1\|, \|F_2\|\} \]

The following theorem is a generalization of Theorem (2) to finite case.

**Theorem (4).** Let \( Z_i, i = 1, 2, ..., n \) be a collection of pair wise closed subsets of a compact metric space \( X \), and let \( f_i: Z_i \rightarrow L_\infty(X): i = 1, 2, ..., n \) be uniformly continuous functions such that \( w_{F_i}(t) \leq w(t) \), where \( w \) is a nondecreasing subadditive uniformly continuous function satisfying \( \lim_{t \to 0} w(t) = 0 \). Then \( f_i : i = 1, 2, 3, ..., n \) can be extended to the same uniformly continuous function \( F: X \rightarrow L_\infty(X) \) whose modulus of continuity \( w_F(t) \leq w(t) + \text{max}_{1 \leq i \leq n}\{|\|F_i\|\} \).

**Proof.** The functions \( f_i : i = 1, 2, 3, ..., n \) are extended to \( F_i: X \rightarrow L_\infty(X); w_{F_i}(t) \leq w(t) \).

By considering each coordinate separately, it suffices to prove this theorem for \( \mathbb{R} \) as the target space instead of \( L_\infty(X) \). Let \( h_j: X \rightarrow [0,1] \) be defined by

\[ h_j(x) = \frac{\pi_{i=1,i\neq j} d(x, Z_i)}{(d(x, Z_j) + \pi_{i=1,i\neq j} d(x, Z_i))} \]

From the definition, \( |h_j| = 1 \), and \( w_{h_j}(t) = \sup\{|h_j(x) - h_j(y)| : d(x, y) \leq t\} \leq 1 \).

Let \( F = \text{max}_{1 \leq i \leq n}\{h_i F_i\} \) be uniformly continuous extension of \( Z_i \) such that \( w_F(t) = w_{\text{max}_{1 \leq i \leq n}\{h_i F_i\}}(t) \leq \text{max}_{1 \leq i \leq n}\{w_{h_i F_i}(t)\} \leq \text{max}_{1 \leq i \leq n}\{|h_i| w_{F_i}(t) + |F_i| w_{h_i}(t)\} \leq w(t) + \text{max}_{1 \leq i \leq n}\{|\|F_i\|\} \).

**Remark (2):** By using Theorem (2) we get that there exist extensions \( F_i, i = 1, 2, ..., n \), such that |\|F_i\|| = |\|f_i\|| and in this case \( w_F(t) \leq w(t) + \text{max}_{1 \leq i \leq n}\{|\|f_i\|\} \).

**References**


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