Inequalities for Modules and
Unitary Invariant Norms

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Abstract. An elementary proof in the case of two operators and forms of the classical inequality of Bohr are presented for linear bounded operators on Hilbert spaces or on pseudo-Hilbert spaces in Lemma 2, Theorem 1 and Theorem 2. We will also investigate some variant of the inequalities of O. T. Pop, see [9], for linear and bounded operators on Hilbert spaces and then for linear and bounded operators with orthogonal ranges. Then several applications of these results for unitary invariant norms and generalized derivative are given in Proposition 5 and 4 and a generalization of Lemma 1 see [6].

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1. Introduction

We recall the classical Bohr's inequality, see [3]. For any $z, w \in \mathbb{C}$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$|z + w|^2 \leq p|z|^2 + q|w|^2,$$

with equality if and only if $w = (p - 1)z$.

Many interesting generalizations of Bohr's inequality have been obtained last years. In this paper we will discuss some forms of the inequality in the case when we have two operators or two positive numbers.

We need the following definition which was presented in [7]. Let $Z$ be an admissible space in the Loynes sense. A linear topological space $\mathcal{H}$ is called pre-Loynes $Z$–space if it satisfies the following properties:

$\mathcal{H}$ is endowed with a $Z$–valued inner product (gramian), i.e. there exists a mapping $\mathcal{H} \times \mathcal{H} \ni (h, k) \rightarrow [h, k] \in Z$ having the properties: $[h, h] \geq 0$;
\[ [h, h] = 0 \text{ implies } h = 0; \ [h_1 + h_2, h] = [h_1, h] + [h_2, h]; \ \lambda h, k = \lambda [h, k]; \]
\[ [h, k]^* = [k, h]; \]
for all \( h, k, h_1, h_2 \in \mathcal{H} \) and \( \lambda \in \mathbb{C} \).

The topology of \( \mathcal{H} \) is the weakest locally convex topology on \( \mathcal{H} \) for which the application \( \mathcal{H} \ni h \to [h, h] \in \mathbb{Z} \) is continuous. Moreover, if \( \mathcal{H} \) is a complete space with this topology, then \( \mathcal{H} \) is called Loynes \( Z \)-space (pseudo-Hilbert space).

Let \( B(\mathcal{H}) \) be the space of linear and bounded operators on Hilbert space \( \mathcal{H} \) and \( B^*(\mathcal{H}) \) the space of linear and bounded operators which admit adjoint on \( \mathcal{H} \), if \( \mathcal{H} \) is a pseudo-Hilbert space.

2. The results

We can prove a similar result to Lemma 2.2, see [10], in pseudo-Hilbert spaces for example or Hilbert spaces for the modulus of a linear bounded operator.

**Lemma 1.** Let \( A, B \in B^*(\mathcal{H}) \) (the space of linear and bounded operators which admit gramian adjoint) if \( \mathcal{H} \) is a Loynes \( Z \)-space (or \( A, B \in B(\mathcal{H}) \) if \( \mathcal{H} \) is a Hilbert space).

If \( |A \pm B| = (|A|^2 + |B|^2)^{\frac{1}{2}} \) then \( |\alpha A \pm \beta B| = (\alpha^2 |A|^2 + \beta^2 |B|^2)^{\frac{1}{2}} \), for all \( \alpha, \beta \) non-negative real numbers.

**Proof.** By \( |A \pm B|^2 = (A^* \pm B^*)(A \pm B) = |A|^2 + |B|^2 \pm B^* A A^* B = |A|^2 + |B|^2 \) it follows that \( B^* A + A^* B = 0 \) and then \( |\alpha A \pm \beta B|^2 = \alpha^2 |A|^2 + \beta^2 |B|^2 \).

We shall give below another proof for a variant of Bohr’s inequality that was given in [12] Theorem 6.

**Lemma 2.** Let \( A, B \in B^*(\mathcal{H}) \), where \( \mathcal{H} \) is a Loynes \( Z \)-space or \( A, B \in B(\mathcal{H}) \) where \( \mathcal{H} \) is a complex separable Hilbert space.

(i) If \( \alpha, \beta, \gamma \) are three real numbers which satisfies the conditions \( \beta \neq 0, \gamma \neq 0 \), \( \alpha + \beta \gamma > 0 \) and \( \alpha > 0 \) or \( \alpha + \beta \gamma < 0 \) and \( \alpha < 0 \) then

\[ |\alpha A - B|^2 + |\beta A - \gamma B|^2 \geq \beta (\beta - \alpha \gamma) |A|^2 + \gamma (\gamma - \frac{\beta}{\alpha}) |B|^2. \]

(ii) If \( \alpha, \beta, \gamma \) are three real numbers which satisfies the conditions \( \alpha \neq 0, \beta \neq 0, \gamma \neq 0 \), \( \alpha + \beta \gamma > 0 \) and \( \frac{\alpha}{\gamma} > 0 \) or \( \alpha + \beta \gamma < 0 \) and \( \frac{\alpha}{\gamma} < 0 \) then

\[ |\alpha A - B|^2 + |\beta A - \gamma B|^2 \geq \alpha (\alpha - \frac{\beta}{\gamma}) |A|^2 + (1 - \frac{\alpha \gamma}{\beta}) |B|^2. \]

(iii) If \( \alpha, \beta, \gamma \) are three real numbers which satisfies the conditions \( \beta \neq 0, \gamma \neq 0 \), \( \alpha + \beta \gamma < 0 \) and \( \alpha > 0 \) or \( \alpha + \beta \gamma > 0 \) and \( \alpha < 0 \) then

\[ |\alpha A - B|^2 + |\beta A - \gamma B|^2 \leq \beta (\beta - \alpha \gamma) |A|^2 + \gamma (\gamma - \frac{\beta}{\alpha}) |B|^2. \]
(iv) If $\alpha$, $\beta$, $\gamma$ are three real numbers which satisfies the conditions $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$, $\alpha + \beta\gamma < 0$ and $\frac{\beta}{\gamma} > 0$ or $\alpha + \beta\gamma > 0$ and $\frac{\beta}{\gamma} < 0$ then

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 \leq \alpha(\alpha - \frac{\beta}{\gamma})|A|^2 + (1 - \frac{\alpha\gamma}{\beta})|B|^2.$$ 

Proof. Using the definition of the modulus of an operator we obtain,

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 = (\alpha^2 + \beta^2)|A|^2 + (\gamma^2 + 1)|B|^2 - (\alpha + \beta\gamma)(B^* A + A^* B) =

= [(\alpha + \beta\gamma)(\alpha + \frac{\beta}{\gamma}) - \alpha\beta(\gamma + \frac{1}{\gamma})]|A|^2 + [(\alpha + \beta\gamma)(\frac{\gamma}{\beta} + \frac{1}{\alpha}) - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})]|B|^2 -

- (\alpha + \beta\gamma)(B^* A + A^* B) = (\alpha + \beta\gamma)(\alpha + \frac{\beta}{\gamma})|A|^2 + (\frac{\gamma}{\beta} + \frac{1}{\alpha})|B|^2 - (B^* A + A^* B) -

- \alpha\beta(\gamma + \frac{1}{\gamma})|A|^2 - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})|B|^2 = (\alpha + \beta\gamma)|\alpha|A|^2 + \frac{\beta}{\gamma}|A|^2 + \frac{\gamma}{\beta}|B|^2 + \frac{1}{\alpha}|B|^2 - (B^* A +

+ A^* B)] - \alpha\beta(\gamma + \frac{1}{\gamma})|A|^2 - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})|B|^2.

For (i) we will consider the following continuation

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 = (\alpha + \beta\gamma)[|\alpha|A|^2 - (B^* A + A^* B)] + \frac{1}{|\alpha|}|B|^2] + \frac{\gamma}{\beta}|A|^2 + |B|^2] -

- \alpha\beta(\gamma + \frac{1}{\gamma})|A|^2 - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})|B|^2 = (\alpha + \beta\gamma)|\sqrt{\alpha} A| - \frac{1}{\sqrt{\alpha}}|B|^2 + [(\alpha + \beta\gamma)\frac{\beta}{\gamma} - \alpha\beta(\gamma + \frac{1}{\gamma})]|A|^2 +

+ [(\alpha + \beta\gamma)\frac{\gamma}{\beta} - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})]|B|^2 \geq \beta(\beta - \alpha\gamma)|A|^2 + \gamma(\gamma - \frac{\beta}{\alpha})|B|^2.

For the second part of (i) we will write

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 = (\alpha + \beta\gamma)[|\alpha|A|^2 - (B^* A + A^* B)] - \frac{1}{|\alpha|}|B|^2] + \frac{\gamma}{\beta}|A|^2 + |B|^2] -

- \alpha\beta(\gamma + \frac{1}{\gamma})|A|^2 - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})|B|^2 = -(\alpha + \beta\gamma)|\sqrt{\alpha} A| + \frac{1}{\sqrt{|\alpha|}}|B|^2 + \beta(\beta - \alpha\gamma)|A|^2 +

+ \gamma(\gamma - \frac{\beta}{\gamma})|B|^2 \geq \beta(\beta - \alpha\gamma)|A|^2 + \gamma(\gamma - \frac{\beta}{\gamma})|B|^2.

For (ii) we have,

$$|\alpha A - B|^2 + |\beta A - \gamma B|^2 = (\alpha + \beta\gamma)[|\alpha|A|^2 + \frac{1}{|\alpha|}|B|^2] + \frac{\beta}{\gamma}|A|^2 - (B^* A + A^* B) + \frac{\gamma}{\beta}|B|^2] -

- \alpha\beta(\gamma + \frac{1}{\gamma})|A|^2 - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})|B|^2 = (\alpha + \beta\gamma)|\sqrt{\beta} A| - \sqrt{\frac{\gamma}{\beta}}|B|^2 + [(\alpha + \beta\gamma)\alpha -

- \alpha\beta(\gamma + \frac{1}{\gamma})|A|^2 + [(\alpha + \beta\gamma)\frac{1}{\alpha} - \gamma(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})]|B|^2 \geq \alpha(\alpha - \frac{\beta}{\gamma})|A|^2 + (1 - \frac{\alpha\gamma}{\beta})|B|^2.

(iii) and (iv) will result from (i) and (ii).
Theorem 1. Let \( A, B \in B^*(H) \), where \( H \) is a Loynes \( Z \)-space or \( A, B \in B(H) \), where \( H \) is a complex separable Hilbert space.

(i) If \( a, b, c, d \in \mathbb{R} \), \( a, b, c, d \neq 0 \) and satisfies the conditions \( ab + cd > 0 \), and \( \frac{a}{b} > 0 \) or \( ab + cd < 0 \), and \( \frac{a}{b} < 0 \) then

\[
|aA - bB|^2 + |cA - dB|^2 \geq c^2(1 - \frac{ad}{cb})|A|^2 + d^2(1 - \frac{cb}{ad})|B|^2,
\]

or if \( a, b, c, d \in \mathbb{R} \), \( a, b, c, d \neq 0 \) and satisfies the conditions \( ab + cd > 0 \) and \( \frac{a}{b} > 0 \) or \( ab + cd < 0 \) and \( \frac{a}{b} < 0 \) then

\[
|aA - bB|^2 + |cA - dB|^2 \geq c^2(1 - \frac{bc}{ad})|A|^2 + b^2(1 - \frac{ad}{bc})|B|^2.
\]

(ii) If \( a, b, c, d \in \mathbb{R} \), \( a, b, c, d \neq 0 \) and satisfies the conditions \( ab + cd < 0 \), and \( \frac{a}{b} > 0 \) or \( ab + cd > 0 \), and \( \frac{a}{b} < 0 \) then

\[
|aA - bB|^2 + |cA - dB|^2 \leq c^2(1 - \frac{ad}{cb})|A|^2 + d^2(1 - \frac{cb}{ad})|B|^2,
\]

or if \( a, b, c, d \in \mathbb{R} \), \( a, b, c, d \neq 0 \) and satisfies the conditions \( ab + cd < 0 \) and \( \frac{a}{b} > 0 \) or \( ab + cd > 0 \) and \( \frac{a}{b} < 0 \) then

\[
|aA - bB|^2 + |cA - dB|^2 \leq a^2(1 - \frac{bc}{ad})|A|^2 + b^2(1 - \frac{ad}{bc})|B|^2.
\]

Proof. We write the expression \( |aA - bB|^2 + |cA - dB|^2 \) as \( b^2(|\frac{A}{B} - B|^2 + |\frac{A}{B} - \frac{a}{b}B|^2) \) and use Theorem 1. \( \square \)

Next theorem is a generalization of Theorem 2.3 [1].

Theorem 2. (i) For any \( r \geq 2 \), \( a, b \in \mathbb{R}^+ \) and \( u, v > 0 \) with the property

\[
uv(u + v)^{r-2} = 1
\]

we have

\[
(1 + \frac{u}{v})a^r + (1 + \frac{v}{u})b^r \geq (ua + vb)^r + \frac{1}{2^{r-2}}|b - a|^r.
\]

(ii) For any \( 1 \leq r \leq 2 \), \( a, b \in \mathbb{R}^+ \) and \( u, v > 0 \) with the property

\[
uv(u + v)^{r-2} = 1
\]

we have

\[
(1 + \frac{u}{v})a^r + (1 + \frac{v}{u})b^r \leq (ua + vb)^r + \frac{1}{2^{r-2}}|b - a|^r.
\]

Proof. Using the fact that the functions \( f(x) = x^r \), \( x \geq 0 \) are superquadratic for \( r \geq 2 \) and subquadratic for \( 0 \leq r \leq 2 \), see [1] we obtain by Lemma 1.2, see [1], for \( 0 \leq \alpha_1 \leq 1 \), \( a, b \geq 0 \) that

\[
\alpha_1 f(a) + (1 - \alpha_1) f(b) - f(\alpha_1 a + (1 - \alpha_1 b)) \geq \alpha_1 f((1 - \alpha_1)|b - a|) + (1 - \alpha_1) f(\alpha_1 |b - a|).
\]

(i) As in the proof of Theorem 2.3 [1] we have

\[
\alpha_1 a^r + \beta_1 b^r \geq (\alpha_1 a + \beta_1 b)^r + \alpha_1 \beta_1 (\beta_1^{r-1} + \alpha_1^{r-1})|b - a|^r,
\]

where \( \beta_1 = 1 - \alpha_1. \)
For $\alpha_1 = \frac{\beta}{\beta + \gamma}$, $\beta_1 = \frac{\gamma}{\beta + \gamma}$, $\beta$, $\gamma > 0$ it results
\[
\frac{\beta}{\beta + \gamma} a^r + \frac{\gamma}{\beta + \gamma} b^r \geq \frac{\beta}{(\beta + \gamma)^{r-1}} \left( \frac{\beta}{\beta + \gamma} a + \frac{\gamma}{\beta + \gamma} b \right)^r + \frac{1}{2r-2} |b-a|^r
\]
Because $\frac{\beta}{\beta + \gamma} = 1 - \frac{\beta}{\beta + \gamma}$ and $\frac{\gamma}{\beta + \gamma}, \frac{\gamma}{\beta + \gamma} \in [0, 1]$ and $r \geq 2$ we have $(\frac{\beta}{\beta + \gamma})^{r-1} + (\frac{\gamma}{\beta + \gamma})^{r-1} \geq 1$, and this implies
\[
\frac{(\beta a^r + \gamma b^r)}{\beta + \gamma} \geq \frac{\beta + \gamma}{(\beta + \gamma)^2} \left( \frac{\beta}{\beta + \gamma} a + \frac{\gamma}{\beta + \gamma} b \right)^r + \frac{1}{2r-2} |b-a|^r
\]
or
\[
\frac{\beta + \gamma}{\beta \gamma} (\beta a^r + \gamma b^r) \geq \frac{\beta + \gamma}{\beta \gamma} (\beta a + \gamma b)^r + \frac{1}{2r-2} |b-a|^r
\]
which leads to
\[
\frac{\beta + \gamma}{\gamma} a^r + \frac{\beta + \gamma}{\beta} b^r \geq \frac{\beta}{(\beta \gamma)^{1/2}(\beta + \gamma)^{-2/2}} a + \frac{\gamma}{(\beta \gamma)^{1/2}(\beta + \gamma)^{-2/2}} b \quad \text{and} \quad \frac{1}{2r-2} |b-a|^r.
\]
Now taking $\gamma = \frac{u}{v} \beta$ where $\beta > 0$ we have $\gamma > 0$, $\frac{\beta}{\gamma} = \frac{v}{u}$ and our inequality becomes
\[
(1 + \frac{u}{v}) a^r + (1 + \frac{v}{u}) b^r \geq \left( \frac{u a + v b}{u + v} \right)^r + \frac{1}{2r-2} |b-a|^r
\]
because
\[
\frac{\beta}{\beta \gamma} = \frac{1}{u + v}, \quad \frac{\gamma}{\beta \gamma} = \frac{v}{u + v} = u
\]
and
\[
\frac{\gamma}{(\beta \gamma)^{1/2}(\beta + \gamma)^{-2/2}} = v
\]
by hypothesis.

For (ii) we will use the same argument taking into account that $\alpha_1^{-1} + \beta_1^{-1} \leq \frac{1}{2r-2}$ if $1 \leq r \leq 2$.

**Corollary 1.** If we take above $u = v(p - 1)$ when $r \geq 2$ and $1 < p \leq 2$ then
\[
pa^r + qb^r \geq \frac{1}{2r-2} ((p-1)a + b)^r + \frac{1}{2r-2} |b-a|^r,
\]
for any $a, b \in \mathbb{R}_+$ and $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** From $u = v(p - 1)$ and $1 < p \leq 2$ it results $u > 0$ and
\[
pa^r + qb^r = (1 + p - 1)a^r + (1 + \frac{1}{p-1})b^r \geq v^r ((p-1)a + b)^r + \frac{1}{2r-2} |b-a|^r.
\]
Replacing $u$ from $u = v(p - 1)$ in $uv(u + v)^{r-2} = 1$ we have $v^2 (p - 1)(v(p - 1) + v)^{r-2} = 1$ or $v^r (p - 1)^{r-2} = 1$ that means $v^r = \frac{1}{(p-1)^{r-2}} \geq \frac{1}{2r-2}$. Thus
\[
pa^r + qb^r = (1 + p - 1)a^r + (1 + \frac{1}{p-1})b^r \geq \frac{1}{2r-2} ((p-1)a + b)^r + \frac{1}{2r-2} |b-a|^r.
\]
\[\square\]
Now we consider \( n \) linear bounded operators having orthogonal ranges as below and we prove the following equality, an analogue result of some results from [9], [4], [5]:

**Theorem 3.** If \( n \in \mathbb{N}, n \geq 2, N_1, N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H}) \) with \( N_i^*N_j = 0, (\forall) i, j = 1, ..., n, i \neq j \), i.e the operators have orthogonal ranges, and \( a_1, a_2, ..., a_n \in \mathbb{R} \setminus \{0\} \), then

\[
\frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} |a_i N_j - a_j N_i|^4 = \frac{|N_1 + ... + N_n|^4}{a_1^2 + ... + a_n^2} + \sum_{k=1}^n \left( \frac{1}{a_k^2} - \frac{2}{a_1^2 + ... + a_n^2} \right) |N_k|^4 = \frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} \frac{|a_i N_j + a_j N_i|^4}{a_i^2 a_j^2}.
\]

**Proof.** Taking into account that the operators have orthogonal ranges and using then the previous theorem with \( a_i^2 \) instead of \( a_i \) and \( |N_i|^2 \) instead \( N_i \), \( i = 1, ..., n \), we have

\[
\frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} |a_i N_j - a_j N_i|^4 = \frac{1}{a_1^2 + ... + a_n^2} \sum_{1 \leq i < j \leq n} \left( \frac{a_i^2 |N_j|^2 + a_j^2 |N_i|^2}{a_i^2 a_j^2} \right)^2 = \left( \frac{|N_1|^2 + ... + |N_n|^2}{a_1^2 + ... + a_n^2} \right)^2 + \sum_{k=1}^n \left( \frac{1}{a_k^2} - \frac{2}{a_1^2 + ... + a_n^2} \right) |N_k|^4 = \frac{|N_1 + ... + N_n|^4}{a_1^2 + ... + a_n^2} + \sum_{k=1}^n \left( \frac{1}{a_k^2} - \frac{2}{a_1^2 + ... + a_n^2} \right) |N_k|^4.
\]

\[\square\]

As a consequence of the Theorem 3 and Proposition 2.7, see [8], we have:

**Consequence 1.** If \( n \in \mathbb{N}, n \geq 2, N_1, N_2, ..., N_n \in \mathcal{B}^*(\mathcal{H}), \) where \( \mathcal{H} \) is a Loynes \( Z \)- space with \( N_i^*N_j = 0, (\forall) i, j = 1, ..., n, i \neq j \), i.e the operators have orthogonal ranges and \( a_1, a_2, ..., a_n, \alpha_1, ..., \alpha_n \in R_+ \setminus \{0\} \) then,

\[
\sum_{1 \leq i < j \leq n} \frac{|\sqrt{a_i} N_j - \sqrt{a_j} N_i|^4}{a_i a_j} \leq \sum_{i=1}^n \frac{1}{\alpha_i} \sum_{k=1}^n a_k + \frac{n}{\alpha_i} \sum_{k=1}^n a_k - 2 |N_i|^4.
\]

**Proof.** Taking in Proposition 2.7 \( r = 4 \) we have

\[
|N_i|^4 \leq \sum_{i=1}^n \frac{1}{\alpha_i} \sum_{i=1}^n \alpha_i |N_i|^4.
\]

Using now Theorem 3 we obtain:

\[
\sum_{1 \leq i < j \leq n} \frac{|\sqrt{a_i} N_j - \sqrt{a_j} N_i|^4}{a_i a_j} \leq \sum_{i=1}^n \frac{1}{\alpha_i} \sum_{i=1}^n \alpha_i |N_i|^4 + \sum_{k=1}^n \left( \frac{1}{\alpha_k} \sum_{i=1}^n a_i - 2 \right) |N_k|^4 =
\]
Inequalities for modules and unitary invariant norms

\[ \sum_{i=1}^{n} \left( \alpha_i \sum_{k=1}^{n} \frac{1}{a_k} \sum_{k=1}^{n} a_k - 2 \right) |N_i|^4. \]

\[ \square \]

Let \( x = (x_1, ..., x_n) \in \mathbb{R}^n \), and we define, see [5], an \( n \times n \) matrix \( \Lambda(x) = x^*x = (x_ix_j) \) and \( D(x) = \text{diag}(x_1, ..., x_n) \). Now we will give an analogue of Theorem 3.1 from [5] for operators with orthogonal ranges.

**Proposition 1.** If \( \Lambda(a^2) + \Lambda(b^2) \leq D(c^2) \) for \( a, b, c \in \mathbb{R}^n \), then

\[ |\sum_{i=1}^{n} a_iA_i|^4 + |\sum_{i=1}^{n} b_iA_i|^4 \leq \sum_{i=1}^{n} c_i|A_i|^4 \]

for arbitrary \( n \)-tuple \((A_i)\) of operators with orthogonal ranges (i.e. \( A_j^*A_i = 0 \), \( \forall i \neq j, i,j \in \{1, ..., n\} \)) in \( \mathcal{B}^*(\mathcal{H}) \) or \( \mathcal{B}(\mathcal{H}) \) respectively. In addition, if \( \Lambda(a^2) + \Lambda(b^2) \geq D(c^2) \) for \( a, b, c \in \mathbb{R}^n \), then

\[ |\sum_{i=1}^{n} a_iA_i|^4 + |\sum_{i=1}^{n} b_iA_i|^4 \geq \sum_{i=1}^{n} c_i|A_i|^4 \]

for arbitrary \( n \)-tuple \((A_i)\) of operators with orthogonal ranges in \( \mathcal{B}^*(\mathcal{H}) \) or \( \mathcal{B}(\mathcal{H}) \) respectively.

**Proof.** We take into account that

\[ |\sum_{i=1}^{n} a_iA_i|^4 + |\sum_{i=1}^{n} b_iA_i|^4 = |\sum_{i=1}^{n} a_i^2|A_i|^2|^2 + |\sum_{i=1}^{n} b_i^2|A_i|^2|^2 \]

and then use Theorem 3.1, [5]. \[ \square \]

Using Theorem 28 from [2] we have:

**Proposition 2.** Let \( A_i \in \mathcal{B}^*(\mathcal{H}) \) with \( A_j^*A_i = 0 \), \( 1 \leq i \neq j \leq n \) and \( \alpha_{ik}, p_i \in \mathbb{R} \) for \( i = 1, 2, ..., n \) and \( k = 1, 2, ..., m \). Define \( X = (x_{ij}) \) where \( x_{ij} = \sum_{k=1}^{m} \alpha_{ik}^4 - p_i \) if \( i = j \) and \( x_{ij} = \sum_{k=1}^{m} \alpha_{ik}^2 \alpha_{jk}^2 \) if \( i \neq j \).

If \( X \geq 0 \) then

\[ \sum_{k=1}^{m} |\sum_{i=1}^{n} \alpha_{ik}A_i|^4 \geq \sum_{i=1}^{n} p_i|A_i|^4. \]

If \( X \leq 0 \) then

\[ \sum_{k=1}^{m} |\sum_{i=1}^{n} \alpha_{ik}A_i|^4 \leq \sum_{i=1}^{n} p_i|A_i|^4. \]

**Proof.** We use that

\[ \sum_{k=1}^{m} |\sum_{i=1}^{n} \alpha_{ik}A_i|^4 = \sum_{k=1}^{m} \left( \sum_{i=1}^{n} \alpha_{ik}^2|A_i|^2 \right)^2 \geq \sum_{i=1}^{n} p_i|A_i|^4, \]

if \( X \geq 0 \), i.e. use Theorem 28, [2]. \[ \square \]
Proposition 3. If \( n \in \mathbb{N}, n \geq 2, N_1, N_2, \ldots, N_n \in \mathcal{B}(\mathcal{H}) \) and \( \alpha_1, \alpha_2, \ldots, \alpha_n \in R \setminus \{0\} \) with \( \alpha_1 + \alpha_2 + \ldots + \alpha_n \neq 0 \) and \( \Lambda(a) + \Lambda(b) \leq D(c) \) for \( a, b, c \in \mathbb{R}^n \) then
\[
\sum_{1 \leq i < j \leq n} \frac{|\alpha_i a_j N_j - \alpha_j a_i N_i|^2}{\alpha_i \alpha_j} + \frac{|\alpha_i b_j N_j - \alpha_j b_i N_i|^2}{\alpha_i \alpha_j} \geq \sum_{i=1}^{n} \left( \frac{a_i + b_i}{\alpha_i} \sum_{k=1}^{n} \alpha_k c_k \right) |N_i|^2.
\]

As in the case of real and complex numbers, see [11] it can be shown the following two inequalities, if we take into consideration \( n = 2 \):

Remark 1. For any linear gramian bounded operators \( N_1, N_2 \in \mathcal{B}^*(\mathcal{H}) \) we have
\[
(2) \quad \frac{|N_1 + N_2|^2}{u + v} \leq \frac{|N_1|^2}{u} + \frac{|N_2|^2}{v},
\]
if \( u, v \neq 0, u + v \neq 0, uv(u + v) > 0 \),
and
\[
(3) \quad \frac{|N_1 + N_2|^2}{u + v} \geq \frac{|N_1|^2}{u} + \frac{|N_2|^2}{v},
\]
if \( u, v \neq 0, u + v \neq 0, uv(u + v) < 0 \).

In addition, if \( N_1^* N_2 = 0 \), then
\[
(4) \quad \frac{|N_1|^4}{u} + \frac{|N_2|^4}{v} \geq \frac{|N_1 + N_2|^4}{u + v},
\]
if \( u, v \neq 0, u + v \neq 0, uv(u + v) > 0 \),
or
\[
(5) \quad \frac{|N_1 + N_2|^4}{u + v} \geq \frac{|N_1|^4}{u} + \frac{|N_2|^4}{v},
\]
if \( u, v \neq 0, u + v \neq 0, uv(u + v) < 0 \).

We can obtain below another generalization of a theorem presented in [9].

Remark 2. (i) If \( n \in \mathbb{N}, n \geq 2, N_i \in \mathcal{B}^*(\mathcal{H}), i \in \overline{1, n} \) and \( a_1, a_2, \ldots, a_n \in R \setminus \{0\} \) with \( a_1 + a_2 + \ldots + a_n \neq 0 \), and \( a_m(a_k + a_l) > 0, (\forall) m = \overline{1, n}, m \neq k, l, a_k, a_l \neq 0 \),
then
\[
\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + \ldots + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + \ldots + N_n|^2}{a_1 + a_2 + \ldots + a_n} \geq N_{k,l} + \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leq i < j \leq n, i,j \neq k,l} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j},
\]
where
\[
N_{k,l} = \max_{1 \leq i < j \leq n} \frac{|a_i N_j - a_j N_i|^2}{a_i a_j(a_i + a_j)} = \frac{|a_k N_l - a_l N_k|^2}{a_k a_l(a_k + a_l)}, 1 \leq k < l \leq n,
\]
if there exist.
(ii) If \( n \in \mathbb{N}, n \geq 2, N_i \in \mathcal{B}(\mathcal{H}), i = \overline{1,n} \) and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\} \) with \( a_1 + a_2 + \ldots + a_n \neq 0 \), and

\[
a_m(a_k + a_l) < 0, \quad (\forall) \ m = \overline{1,n}, \ m \neq k, l, \ a_k, a_l \neq 0,
\]

then

\[
\frac{|N_1|^2}{a_1} + \frac{|N_2|^2}{a_2} + \ldots + \frac{|N_n|^2}{a_n} - \frac{|N_1 + N_2 + \ldots + N_n|^2}{a_1 + a_2 + \ldots + a_n} \leq
\]

\[
\leq N_{k,l} + \frac{1}{a_1 + a_2 + \ldots + a_n} \sum_{1 \leq i < j \leq n, i, j \neq k, l} |a_iN_j - a_jN_i|^2 / a_i a_j,
\]

where

\[
N_{k,l} = \min_{1 \leq i < j \leq n} \frac{|a_iN_j - a_jN_i|^2}{a_i a_j(a_i + a_j)}, \ 1 \leq k < l \leq n,
\]

if there exist.

**Proof.** The proof will be as in the case of real numbers, see [9].

\[\square\]

We will present now a generalization of Theorem 4 when the operators \( N_i, i = \overline{1,n} \) have the orthogonal ranges.

**Theorem 4.** If \( n \in \mathbb{N}, n \geq 2, N_i \in \mathcal{B}(\mathcal{H}), i = \overline{1,n} \) with \( N_i^*N_j = 0, \ (\forall) \ i, j = 1, \ldots, n, \ i \neq j \), i.e. the operators have orthogonal ranges and \( a_1, a_2, \ldots, a_n \in \mathbb{R} \setminus \{0\} \) then

\[
\frac{|N_1 + \ldots + N_n|^4}{a_1^2 + \ldots + a_n^2} + \sum_{k=1}^{n} \left( \frac{1}{a_k^2} - \frac{2}{a_k^2 + \ldots + a_n^2} \right) |N_k|^4 \geq
\]

\[
\geq \frac{|a_iN_j - a_jN_i|^4}{a_i^2 a_j^2(a_i^2 + a_j^2)} + \frac{1}{a_i^2 + \ldots + a_n^2} \sum_{1 \leq i < j \leq n} |a_iN_j - a_jN_i|^4 / a_i^2 a_j^2,
\]

where

\[
N_{k,l} = \max_{1 \leq i < j \leq n} \frac{|a_iN_j - a_jN_i|^4}{a_i^2 a_j^2(a_i^2 + a_j^2)} = \frac{|a_kN_l - a_lN_k|^4}{a_k^2 a_l^2(a_k^2 + a_l^2)}, \ 1 \leq k < l \leq n,
\]

if there exist.

**Proof.** The proof will be as in the case of real numbers, see [9], using Remark 2 and Theorem 3.

\[\square\]
Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. We shall consider as in [6], a unitarily invariant norm $|||.|.||$ which is a norm on an ideal $C_{|||.|.||}$ of $\mathcal{B}(\mathcal{H})$, making $C_{|||.|.||}$ into a Banach space and satisfying $|||UXV||| = |||X|||$ for all $X \in \mathcal{B}(\mathcal{H})$ and all unitary operators $U$ and $V$ in $\mathcal{B}(\mathcal{H})$.

**Lemma 3** ([6]). Let $A, B, X \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space with $A$ and $B$ self-adjoint and $X \geq \gamma I$, in which $\gamma$ is a positive real number. Then

$$\gamma |||A - B||| \leq |||AX - BX|||.$$

In the following we shall give some generalizations of Theorem 2 of O. Hirzallah, see [6].

**Theorem 5.** Let $A, B, X \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a complex separable Hilbert space and $X \geq \gamma I$, for positive real number $\gamma$.

If $a, b, c, d \in \mathbb{R}$, $a, b, c, d \neq 0$ and satisfies the conditions $ab + cd < 0$ and $\frac{c}{d} > 0$ or $ab + cd > 0$ and $\frac{c}{d} < 0$ then

$$\gamma |||aA - bB|^2 + |cA - dB|^2||| \leq |||a^2(1 - \frac{bc}{ad})|A|^2X + b^2(1 - \frac{ad}{bc})X|B|^2|||.$$

**Proof.** We replace in Lemma 3, $A$ by $a^2(1 - \frac{bc}{ad})|A|^2$ and $B$ by $-b^2(1 - \frac{ad}{bc})|B|^2$ and we use Theorem 1, (ii). \qed

**Remark 3.** (a) Let $A, B, X \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space as in [6] and $1 + \beta \gamma < 0$. If $X \geq \gamma_1 I$, $\gamma_1$ is a positive real number then

$$\gamma_1 |||A - B|^2 + |\beta A - \gamma B|^2||| \leq |\beta - \gamma_1|||\beta|A|^2X - \gamma X|B|^2|||.$$

(b) If we take in Theorem 3, (i), $a = 1 - p$, $b = c = d = 1$ we obtain

$$\gamma_1 |||(1 - p)A - B|^2 + |A - B|^2||| \leq |||p|A|^2X + qX|B|^2|||,$$

when $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(c) If we take in Theorem 3, (ii), $a = b = c = 1$ and $d = 1 - q$ we obtain

$$\gamma_1 |||A - B|^2 + |A - (1 - q)B|^2||| \leq |||p|A|^2X + qX|B|^2|||,$$

when $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

We recall that the generalized derivation $\delta_{A,B}$ of two operators $A, B \in \mathcal{B}(\mathcal{H})$ is defined by $\delta_{A,B}(X) = AX - XB$ for all $X \in \mathcal{B}(\mathcal{H})$. Also $\delta_{A,B}^2(X) = \delta_{A,B}(\delta_{A,B}(X))$.

We shall give a generalization of Theorem 4 from [6]. The constants $p$ and $q$ from Theorem 4 can be replaced by $a$, $b$, $c$, $d$ as below.
Proposition 4. Let $A, B, X \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a complex separable Hilbert space and $A, B$ are normal operators.

(i) If $a, b, c, d \in \mathbb{R}$, $a, b, c, d \neq 0$ and satisfies the conditions $ab + cd < 0$, and $\frac{c}{b} > 0$ or $ab + cd > 0$, and $\frac{c}{b} < 0$ then
\[
||\delta_{aA,bB}(X)||_2^2 + ||\delta_{cA,dB}(X)||_2^2 \leq ||c^2(1 - \frac{ad}{bc})|A|^2X + d^2(1 - \frac{cb}{ad})|B|^2||_2^2.
\]

(ii) If $a, b, c, d \in \mathbb{R}$, $a, b, c, d \neq 0$ and satisfies the conditions $ab + cd > 0$, and $\frac{c}{b} > 0$ or $ab + cd < 0$, and $\frac{c}{b} < 0$ then
\[
||\delta_{aA,bB}(X)||_2^2 + ||\delta_{cA,dB}(X)||_2^2 \geq \frac{1}{2}||c^2(1 - \frac{ad}{bc})|A|^2X + d^2(1 - \frac{cb}{ad})|B|^2||_2^2.
\]

Proof. It will be as in [6], Theorem 4, but we will use Consequence 3 (ii), (i) instead of inequality (7), see [6]. Also we use the elementary inequalities $(x + y)^2 \leq 2(x^2 + y^2)$ and $x^2 + y^2 \leq (x + y)^2$ for all $x, y \geq 0$.

We know, see [11] for example, that for $uv(u + v) > 0$ and $r > 1$, $z_1, z_2 \in \mathbb{C}$ we have
\[
\frac{|z_1 + z_2|^r}{u + v} \leq \frac{|z_1|^r}{u} + \frac{|z_2|^r}{v}
\]
and for $uv(u + v) < 0$ and $r > 1$, $z_1, z_2 \in \mathbb{C}$ we have
\[
\frac{|z_1 + z_2|^r}{u + v} \geq \frac{|z_1|^r}{u} + \frac{|z_2|^r}{v}.
\]

Considering $r \in \mathbb{N}$, $r \neq 0$ we can define $\delta_{A,B}^r(X)$ by $\delta_{A,B}^r(X) = \delta_{A,B}(\delta_{A,B}^{r-1}(X))$ and by induction we can show that
\[
\delta_{A,B}^r(X) = A^rX - C^1_r A^{r-1}XB + ... + (-1)^{r-1}C^1_r A^{r-1}XB^{r-1} + (-1)^rXB^r.
\]

We will give two similar properties for the generalized derivation $\delta_{A,B}$ of two normal operators $A, B \in \mathcal{B}(\mathcal{H})$.

Proposition 5. Let $A, B, X \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a complex separable Hilbert space and $A, B$ are normal operators.

(i) If $uv > 0$ and $r \geq 1, r \in \mathbb{N}$ then
\[
\frac{1}{(u + v)^2}||\delta_{A,-B}^r(X)||_2^2 \leq \frac{1}{u}||A^rX + \frac{1}{v}X|B|^r||_2^2.
\]
for every $X \in \mathcal{B}(\mathcal{H})$.

(ii) If $r \in \mathbb{N}$, $r \geq 2$ and $w_1, w_2 \in \mathbb{R}_+$ then
\[
||\delta_{A,-B}^r(X)||_2^2 \leq \frac{w_1}{w_1^{1-r} + w_2^{1-r}}||A^rX + \frac{w_2}{w_1^{1-r} + w_2^{1-r}}X|B|^r||_2^2.
\]

(iii) If $r \geq 3$, $r \in \mathbb{N}$ and $D_1, D_2$ are two diagonal positive operators in $\mathcal{B}(\mathcal{H})$ with $u, v$ as in Theorem 2 then
\[
||\delta_{uD_1,-vD_2}^r(X)||_2^2 + \frac{1}{2^{2(r-2)}}||\delta_{D_1,D_2}^r(X)||_2^2 \leq \frac{1}{u}||D_1^rX + (1 + \frac{v}{u})XD_2^r||_2.
\]
Proof. We use the same method as in [6]. We use Voiculescu’s Theorem. Given \( \varepsilon > 0 \) there are diagonal operators \( D_1, D_2 \) and Hilbert-Schmidt operators \( K_1, K_2 \) such that \( A = D_1 + K_1, \ B = D_2 + K_2, \ ||K_1|| < \varepsilon, \ ||K_2|| < \varepsilon, \ D_1 e_i = \lambda_i e_i \) and \( D_2 f_i = \mu_i f_i, \ i \in \mathbb{N} \) for some orthonormal bases \{\( e_i \)\} and \{\( f_i \)\} for \( \mathcal{H} \) and some sequences \{\( \lambda_i \)\} and \{\( \mu_i \)\} of complex numbers. If we show that the inequality is true for \( D_1, D_2 \) then by a limit argument the inequality from (i), (ii) will be true.

For (i), we calculate

\[
\frac{1}{(u+v)^2} ||\delta_{D_1,-D_2}(X)||^2 = \frac{1}{(u+v)^2} \sum_{i,j=1}^{\infty} | < \delta_{D_1,-D_2}(X)f_j, e_i > |^2 = \\
= \sum_{i,j=1}^{\infty} \left( \frac{\lambda_i + \mu_j}{u+v} \right)^2 | Xf_j, e_i > |^2 = \sum_{i,j=1}^{\infty} \left( \frac{\lambda_i}{u} + \frac{\mu_j}{v} \right)^2 | Xf_j, e_i > |^2 = \\
= \left| \frac{1}{u} |D_1|^r X + \frac{1}{v} X |D_2|^r \right|^2.
\]

For (ii), we have

\[
||\delta_{D_1,-D_2}(X)||^2 = \sum_{i=1}^{\infty} | < \delta_{D_1,-D_2}(X)f_j, e_i > |^2 = \sum_{i,j=1}^{\infty} | \lambda_i + \mu_j |^2 | Xf_j, e_i > |^2 \leq \\
\leq \sum_{i,j=1}^{\infty} \left( \frac{w}{w_1 + w_2} |\lambda_i|^r + \frac{w_2}{w_1 + w_2} |\mu_j|^r \right)^2 | Xf_j, e_i > |^2 = \\
= \left| \frac{w_1}{w_1 + w_2} |D_1|^r X + \frac{w_2}{w_1 + w_2} X |D_2|^r \right|^2,
\]

taking into account that we used inequality (1.1) from Theorem 1, see [11].

For (iii), we use Theorem 2 (i) and we obtain:

\[
||\delta_{uD_1,-vD_2}(X)||^2 + \frac{1}{2^{2(r-2)}} ||\delta_{D_1,D_2}(X)||^2 = \sum_{i,j=1}^{\infty} | < \delta_{uD_1,-vD_2}(X)f_j, e_i > |^2 + \\
+ \frac{1}{2^{2(r-2)}} \sum_{i,j=1}^{\infty} | < \delta_{D_1,D_2}(X)f_j, e_i > |^2 = \sum_{i,j=1}^{\infty} | u\lambda_i + v\mu_j |^r Xf_j, e_i > |^2 + \\
+ \frac{1}{2^{2(r-2)}} \sum_{i,j=1}^{\infty} | < \lambda_i - \mu_j |^r Xf_j, e_i > |^2 = \sum_{i,j=1}^{\infty} \left( (u\lambda_i + v\mu_j)^{2r} + \frac{1}{2^{2(r-2)}} |\lambda_i - \mu_j|^{2r} \right) \\
| < Xf_j, e_i > |^2 \leq \sum_{i,j=1}^{\infty} \left( (u\lambda_i + v\mu_j)^{2r} + \frac{1}{2^{2(r-2)}} |\lambda_i - \mu_j|^{2r} \right) | Xf_j, e_i > |^2 \leq \\
\leq \sum_{i,j=1}^{\infty} \left( (1 + \frac{u}{v})\lambda_i^r + (1 + \frac{v}{u})\mu_j^r \right)^2 | Xf_j, e_i > |^2 = \left| \frac{1}{u} |D_1|^r X + \frac{1}{v} X |D_2|^r \right|^2.
\]
We also can replace in Theorem 3 from [6] the constants p and q with c and d as below.

**Proposition 6.** Let $A, B, X \in B(H)$, where $H$ is a complex separable Hilbert space and $c, d \in \mathbb{R} - \{0\}$ with $cd < -1$. If $X \in B(H)$ with $X \geq (c - d)(c|A|^2 - d|B|^2)$, then

$$|||A - B|||_4 \leq |c - d| \cdot ||c|A|^2 X - dX|B|^2||.$$