Fixed Point Theorems of Self Mapping

in a Complete 2 Metric Spaces

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Abstract

In this paper, we have obtained some fixed point theorems in 2-metric spaces using mixed type of contraction mappings [8]. The results presented substantially improve and extend the results due to Saha and Day [9] and Rhoades [6], and our results also generalizes an existing result in 2-metric spaces.

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1. Introduction

The concept of 2-metric space was initiated by S. Gahler [1,2]. The study was further enhanced by B.E. Rhoades [6], Iseki [3], Miczko and Palezewski [4]
and Saha and Day [7], Khan [5]. Moreover B.E. Rhoades and other introduced several properties of 2-metric spaces and proved some fixed point and common fixed point theorems for contractive and expansion mappings and also have found some interesting results in 2-metric space, where in each cases the idea of convergence of sum of a finite or infinite series of real constants plays a crucial role in the proof of fixed point theorems. In this same way, we prove a fixed point theorem and common fixed point theorems for the mapping satisfying different types of contractive conditions in 2-metric space.

2. Definitions and Preliminaries

Definition 2.1
Let X be a non empty set. A real valued function d on X×X×X is said to be a 2-metric in X if
(i) To each pair of distinct points x, y in X. There exists a point z ∈ X such that d(x, y, z) ≠ 0
(ii) d(x, y, z) = 0, when at least of x, y, z are equal.
(iii) d(x, y, z) = d(y, z, x) = d(x, z, y)
(iv) d(x, y, z) ≤ d(x, y, w) + d(x, w, z) + d(w, y, z) for all x, y, z, w ∈ X
When d is a 2-metric on X, then the ordered pair (X,d) is called is 2-metric space.

Definition 2.2
A sequence {x_n} in 2-metric space (X,d) is said to be convergent to an element x ∈ X if
\[ \lim_{n\to\infty} d(x_n, x, a) = 0 \] for all a ∈ X.
It follows that if the sequence {x_n} converges to x then
\[ \lim_{n\to\infty} d(x_n, a, b, c) = d(x, a, b) \] for all a, b ∈ X.

Definition 2.3
A sequence \{x_n\} in a 2-metric space X is a Cauchy sequence if d(x_m,x_n,a)=0 as m,n→∞ for all a ∈ X.

Definition 2.4
If a sequence is convergent in a 2-metric space then it is a Cauchy sequence.

Definition 2.5
A 2-metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Proposition 2.6
If a sequence \{x_n\} in a 2-metric space converges to x then every subsequence of \{x_n\} also converges to the same limit x.
**Proposition 2.7**

Limit of a sequence in a 2-metric space, if exists, is unique.

### 3. Main Results

**Theorem 3.1**

Let \((X,d)\) be a complete 2-metric space. Let \(T\) be a self map on \(X\) satisfying conditions:

\[
d(T^i x, T^i y, a) \leq \beta_i \left[ d(y, Ty, a) + d(y, Tx, a) \right] + \gamma_i d(x, y, a)
\]

for all \(x, y, a \in X\) and

Let \(0 \leq \beta_i, 0 \leq \gamma_i < 1, (i = 1, 2, \ldots)\) with

\[
\sum_{i=1}^{\infty} (\beta_i + \gamma_i) < \infty
\]

Then \(T\) has a unique fixed point in \(X\).

**Proof:**

Let any \(x \in X\), Let \(x_n = T^n(x_0)\)

Then

\[
d(Tx_0, T^2x_0, a) = d(Tx_0 TTx_0, a)
\]

\[
\leq \beta_1 d(Tx_0, TTx_0, a) + d(Tx_0, Tx_0, a) + \gamma_1 d(x_0, Tx_0, a)
\]

\[
= \beta_1 d(Tx_0, T^2x_0, a) + d(x_1, x_1, a) + \gamma_1 d(x_0, Tx_0, a)
\]

\[
\therefore d(Tx_0, T^2x_0, a) \leq \beta_1 d(Tx_0, T^2x_0, a) + \gamma_1 d(x_0, Tx_0, a)
\]

\[
\Rightarrow d(Tx_0, T^2x_0, a) \leq \gamma_1 d(x_0, Tx_0, a)
\]

\[
\Rightarrow d(Tx_0, T^2x_0, a) \leq \frac{\gamma_1}{1 - \beta_1} d(x_0, Tx_0, a)
\]

(2)

Now

\[
d(x_n, x_{n+1}, a) = d(T^n x_0, T^{n+1}x_0, a)
\]

\[
= d(T^nx_0, T^{n+1}x_0, a)
\]

\[
\leq \beta_n d(Tx_0, T^nTx_0, a) + d(Tx_0, Tx_0, a) + \gamma_n d(x_0, Tx_0, a)
\]

\[
d(x_n, x_{n+1}, a) \leq \beta_n d(Tx_0, T^nTx_0, a) + \gamma_n d(x_0, Tx_0, a)
\]

\[
d(x_n, x_{n+1}, a) \leq \beta_n d(Tx_0, T^2x_0, a) + \gamma_n d(x_0, Tx_0, a)
\]

\[
\leq \beta_n \left( \frac{\gamma_1}{1 - \beta_1} \right) d(x_0, Tx_0, a) + \gamma_n d(x_0, Tx_0, a)
\]

by (2)

\[
= \left\{ \left( \frac{\gamma_1}{1 - \beta_1} \right) \beta_n + \gamma_n \right\} d(x_0, Tx_0, a)
\]

Implies that

\[
d(x_n, x_{n+1}, a) \leq \left\{ \left( \frac{\gamma_1}{1 - \beta_1} \right) \beta_n + \gamma_n \right\} d(x_0, Tx_0, a)
\]

(3)
then
\[
d(x_n, x_{n+2}, a) = d(x_n, x_n, a) \\
\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+2}) \\
= d(x_n, x_{n+1}) + \sum_{k=0}^{1} d(x_{n+k}, x_{n+k+1}, a)
\]

Similarly
\[
d(x_n, x_{n+3}, a) \leq \sum_{k=0}^{1} d(x_{n+k}, x_{n+k+1}, a) + \sum_{k=0}^{2} d(x_{n+k}, x_{n+k+1}, a)
\]

So for any positive integer \( p \),
\[
d(x_n, x_{n+p}, a) = \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a)
\]

We have
\[
\sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) = \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a) + \sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1})
\]

\[
\leq \left\{ \left[ \frac{\gamma_1}{1-\beta_1} \right] \beta_n + \gamma_n \right\} d(x_0, x_{n+p}) + \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a)
\]

by (3)

Now
\[
d(x_0, x_{n+p}) = \sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) = 0
\]

Then from (4), we have
\[
d(x_n, x_{n+p}, a) \leq \sum_{k=0}^{p-1} d(x_{n+k}, x_{n+k+1}, a)
\]

\[
\leq \left\{ \left[ \frac{\gamma_1}{1-\beta_1} \right] \beta_n + \gamma_n \right\} d(x_0, x_{n+p}) + \gamma_1 d(x_{n+p}, x_{n+p})
\]

by (3)

So
\[
\sum_{k=0}^{p-2} d(x_{n+p}, x_{n+k}, x_{n+k+1}) = 0
\]
Since $\sum_n (\beta_n + \gamma_n) < \infty$, $d(x_{n+p}, x_n, a) \to 0$ as $n \to \infty$

So $\{x_n\}$ is a Cauchy sequence in $X$ and by completeness of $X$, $x_n \to u \in X$

Again

\[
d(x_{n+1}, Tu, a) = d(T^{n+1}x_0, Tu, a)
= d(TT^nx_0, Tu, a)
\leq \beta_1[d(u, Tu, a) + d(u, TT^n x_0, a)] + \gamma_1 d(T^n x_0, u, a)
= \beta_1[d(u, Tu, a) + d(u, T^{n+1} x_0, a)] + \gamma_1 d(T^n x_0, u, a)
= \beta_1[d(u, Tu, a) + d(u, x_{n+1}, a)] + \gamma_1 d(x_n, u, a)
\]

$\therefore$ $d(x_{n+1}, Tu, a) \leq \beta_1 d(u, Tu, a) + \beta_1 d(u, x_{n+1}, a) + \gamma_1 d(x_n, u, a)$

Taking limit as $n \to \infty$

$\lim_{n \to \infty} d(u, Tu, a) \leq \beta_1 d(u, Tu, a)$

$\Rightarrow Tu = u$

Thus $u$ is a fixed point of $T$

**Uniqueness**

Let $u, v$ be two fixed points of $T$.

$\lim_{n \to \infty} d(u, v, a) = d(Tu, Tv, a)$

$\leq \beta_1 d(v, Tu, a) + \beta_1 d(v, Tu, a) + \gamma_1 d(u, v, a)$

$\lim_{n \to \infty} d(u, v, a) \leq \gamma_1 \lim_{n \to \infty} d(u, v, a)$

Since $0 \leq \gamma_1 < 1$

Thus $u = v$

$\therefore$ we can say that fixed point of $T$ is unique.

**Theorem 3.2**

Let $(X, d)$ be a complete 2-metric space, let $T$ be a self mapping on $X$ satisfying the conditions
\[ d(T^i x, T^i y, a) \leq \beta_i \left[ d(y, Ty, a) + d(y, Tx, a) \right] + \gamma_i d(x, y, a) \]

for all \( x, y, a \in X \); and \( 0 \leq \beta_i, \gamma_i < 1 \), \( (i = 1, 2, \ldots) \) with \( \sum_n (\beta^n + \gamma^n) < \infty \). If for some \( x \in X \), \( \{T^n(x)\} \) has a subsequence \( \{T^{n_k}(x)\} \) with \( \lim_{k \to \infty} T^{n_k}(x) = u \in X \), then \( u \) is the unique fixed point of \( T \).

**Proof.** We have for \( u, x, a \in X \).

\[
d(u, Tu, a) \leq d(u, Tu, T^{n_k+1} x) + d(u, T^{n_k+1} x, a) + d(T^{n_k+1} x, Tu, a)
\]

Taking limit as \( k \to \infty \), we get

\[ d(u, Tu, a) \leq \beta_1 \left[ d(u, Tu, a) + d(u, T^{n_k+1} x, a) \right] + \gamma_1 d(T^{n_k} x, u, a) \]

implies \( d(u, Tu, a) = 0 \)

So \( u = Tu \) and uniqueness of \( u \) is also very clear.

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