On Edge-Balance Index Sets of \((p, p + 1)\)-Graphs

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Abstract

Let \(G\) be a simple graph with vertex set \(V(G)\) and edge set \(E(G)\), and let \(\mathbb{Z}_2 = \{0, 1\}\). Any edge labeling \(f\) induces a partial vertex labeling \(f^+ : V(G) \to \mathbb{Z}_2\) assigning 0 or 1 to \(f^+(v)\), \(v\) being an element of \(V(G)\), depending on whether there are more 0-edges or 1-edges incident with \(v\), and no label is given to \(f^+(v)\) otherwise. For each \(i \in \mathbb{Z}_2\), let \(v_f(i) = |\{v \in V(G) : f^+(v) = i\}|\) and let \(e_f(i) = |\{e \in E(G) : f(e) = i\}|\). An edge-labeling \(f\) of \(G\) is said to be edge-friendly if \(|e_f(0) - e_f(1)| \leq 1\). The edge-balance index set of the graph \(G\) is defined as \(\text{EBI}(G) = \{|v_f(0) - v_f(1)| : f \text{ is edge-friendly.}\}\). In this paper, we investigate and present results concerning the edge-balance index sets of \((p, p + 1)\)-graphs.

Mathematics Subject Classification: 05C78, 05C25

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1 Introduction

In [5], Kong and Lee considered a new labeling problem of graph theory. Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0, 1\}$. An edge labeling $f : E(G) \to \mathbb{Z}_2$ induces a vertex partial labeling $f^+ : V(G) \to \mathbb{Z}_2$ defined by $f^+(v) = 0$ if the edges labeled 0 incident on $v$ is more than the number of edges labeled 1 incident on $v$, and $f^+(v) = 1$ if the edges labeled 1 incident on $v$ is more than the number of edges labeled 0 incident on $v$. $f^+(v)$ is not defined if the number of edges labeled by 0 is equal to the number of edges labeled by 1. For $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$, and let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$.

With this notation, we now introduce the notion of an edge-balanced graph.

**Definition 1.** An edge labeling $f$ of a graph $G$ is said to be **edge-friendly** if $|e_f(0) - e_f(1)| \leq 1$. A graph $G$ is said to be an **edge-balanced** graph if there is an edge-friendly labeling $f$ of $G$ satisfying $|v_f(0) - v_f(1)| \leq 1$.

Chen, Lee, et al in [1] proved that all connected simple graphs except the star $K_{1,2k+1}$, where $k \geq 0$ are edge-balanced.

**Definition 2.** The **edge-balance index set** of the graph $G$, $EBI(G)$, is defined as $\{ |v_f(0) - v_f(1)| :$ the edge labeling $f$ is edge-friendly $\}$. We will use $v(0)$, $v(1)$, $e(0)$, $e(1)$ instead of $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$, provided there is no ambiguity.

**Example 1.** $EBI(nK_2)$ is $\{0\}$ if $n$ is even and $\{2\}$ if $n$ is odd.

\[
\begin{array}{cc}
0 & 0 \\
\hline
1 & 1 \\
\end{array}
\]

$|v(0) - v(1)| = 0$

\[
\begin{array}{cc}
0 & 0 \\
\hline
1 & 1 \\
\end{array}
\]

$|v(0) - v(1)| = 2$

Figure 1: The edge-balance index set of $2K_2$ and $3K_2$

For any $n \geq 1$, we denote the tree with $n + 1$ vertices of diameter two by $St(n)$. The star has a center $c$ and $n$ appended edges from $c$.

**Example 2.** The edge-balance index set of the star $St(n)$ is

$$EBI(St(n)) = \begin{cases} 
\{0\} & \text{if } n \text{ is even}, \\
\{2\} & \text{if } n \text{ is odd}.
\end{cases}$$
**Example 3.** In [14], Lee, Lo and Tao showed that

$$EBI(P_n) = \begin{cases} 
\{2\} & \text{if } n \text{ is 2}, \\
\{0\} & \text{if } n \text{ is 3}, \\
\{1, 2\} & \text{if } n \text{ is 4}, \\
\{0, 1\} & \text{if } n \text{ is odd and greater than 3}, \\
\{0, 1, 2\} & \text{if } n \text{ is even and greater than 4}.
\end{cases}$$

Figure 2 shows the EBI of $P_3$ and $P_4$.

![Figure 2: The edge-balance index set of $P_3$ and $P_4$](image)

<table>
<thead>
<tr>
<th>$v(0) - v(1)$</th>
<th>0</th>
<th>1</th>
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<tbody>
<tr>
<td>$</td>
<td>v(0) - v(1)</td>
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**Example 4.** Figure 3 shows that the edge-balance index set of a tree with six vertices is $\{0, 1, 2\}$.

![Figure 3: The edge-balance index set of a tree with six vertices](image)

<table>
<thead>
<tr>
<th>$v(0) - v(1)$</th>
<th>0</th>
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<td>v(0) - v(1)</td>
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The edge-balance index sets of fans, wheels, product graphs, generalized theta graphs and trees were considered in [2, 3, 7, 9, 14]. In this paper, exact values of the edge-balance index sets of seven families of $(p, p + 1)$-graphs are presented. The balance index set of these seven families can be found in [16].
2 Edge-balance index sets of a cycle with a chord

Notation 1. Let $C_n$ be a cycle with a vertex set $\{c_1, c_2, \cdots, c_n\}$. We denote by $C_n(t)$ a cycle with a chord $(c_1, c_t)$. Also we denote by $\bigcup_{i=1}^k C_{n_i}$ the disjoint union of cycles, where $n_i \geq 3$ for all $i = 1, 2, \cdots, k$.

Notation 2. For $C_n$ or $\bigcup_{i=1}^k C_{n_i}$, we denote the numbers of edges labeled 0 or 1 by $f$ by $e_C(0)$ or $e_C(1)$, respectively. We also denote the number of vertices labeled 0, 1, or not labeled by $f^+$ by $v_C(0)$, $v_C(1)$, or $v_C(\times)$, respectively.

For a vertex of order 2, with an edge labeling (not necessarily edge-friendly), it can only be labeled in one of the following three ways.

1. If both edges are labeled 0, then the vertex is labeled 0.
2. If both edges are labeled 1, then the vertex is labeled 1.
3. If one edge is labeled 0 and another is labeled 1, then the vertex is not labeled.

If we add an edge to a vertex, then there are two cases:

A If the vertex was already labeled, then the label of the vertex is not changed after adding an edge because at least two edges are labeled by the same number.

B If the vertex was not labeled then the label of the vertex is the same as the label assigned to the new edge.

For later reference, we call these as Rule A and Rule B.

Lemma 2.1 Let $f$ be an edge labeling (not necessarily edge-friendly) on a disjoint union of cycles $\bigcup_{i=1}^k C_{n_i}$ where $n_i \geq 3$ for all $i = 1, 2, \cdots, k$. Then, we have two equations:

$$2v_C(0) + v_C(\times) = 2e_C(0)$$

and

$$2v_C(1) + v_C(\times) = 2e_C(1).$$
Proof. Every vertex labeled 1 has two incident 1-edges and every unlabeled vertex has one incident 1-edge. No other vertex contains any edge labeled 1. Because every edge is counted twice, we have
\[ 2v_C(1) + v_C(\times) = 2e_C(1). \]
Similarly, we have
\[ 2v_C(0) + v_C(\times) = 2e_C(0). \]

2

Theorem 2.2 The edge-balance index set of a disjoint union of cycles is
\[ EBI \left( \bigcup_{i=1}^{k} C_{n_i} \right) = \begin{cases} \{0\} & \text{if } \sum_{i=1}^{k} n_i \text{ is even,} \\ \{1\} & \text{if } \sum_{i=1}^{k} n_i \text{ is odd,} \end{cases} \]
where \( n_i \geq 3 \) for all \( i = 1, 2, \ldots, k \).

Proof. For an edge-friendly labeling \( f \), we have \( |e_C(0) - e_C(1)| \leq 1 \). By Lemma 2.1, the edge-balance index is \( |v_C(0) - v_C(1)| = |e_C(0) - e_C(1)| \). This completes the proof. \( \square \)

Corollary 2.3 For \( n \geq 3 \), the edge-balance index set of a cycle \( C_n \) is
\[ EBI(C_n) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases} \]

Theorem 2.4 For any \( t \geq 3 \), the edge-balance index set of \( C_n(t) \) is
\[ EBI(C_n(t)) = \begin{cases} \{0, 1\} & \text{if } n \text{ is odd,} \\ \{0, 1, 2\} & \text{if } n \text{ is even.} \end{cases} \]

Proof. If \( n \) is odd, then the number of edges of \( C_n(t) \) is \( n + 1 \) which is even. Notice that if an edge labeling on \( C_n(t) \) is edge-friendly, then \( e(0) \) equals \( e(1) \). Thus, after removing the chord, the remaining labeling is still edge-friendly since \( |e_C(0) - e_C(1)| = 1 \) in \( C_n \) no matter the removed edge is labeled 0 or 1. Therefore, we get a new graph \( C_n \) with an edge-friendly labeling. Without loss of generality, we assume that \( e_C(0) = e_C(1) + 1 \). Thus, the chord must be labeled 1. From Corollary 2.3, we know that the edge-balance index set of \( C_n \) is \( \{1\} \) since \( n \) is odd. Moreover, since \( e_C(0) = e_C(1) + 1 \), we have \( v_C(0) = v_C(1) + 1 \).

When we put the chord back, we have three possible cases:
1. If the labels of \( c_1 \) and \( c_t \) are the same, then the chord does not change the labels of \( c_1 \) and \( c_t \) because of Rule A. The edge-balance index remains 1.

2. If one of the \( c_1 \) or \( c_t \) is labeled and another one is not labeled, then the unlabeled one becomes the same label as the chord because of Rule B and the labeled vertex remains unchanged because of Rule A. The edge-balance index becomes 0 since the chord is labeled 1.

3. If the \( c_1 \) and \( c_t \) are both not labeled, then the chord changes both labels of \( c_1 \) and \( c_t \) to the label of the chord. Thus, the labels of both \( c_1 \) and \( c_t \) become 1 since the chord is labeled 1. The edge-balance index remains 1 since now \( v(1) = v(0) + 1 \).

Therefore, the edge-balance index set of \( C_n(t) \) is \( \{0, 1\} \) if \( n \) is odd. If \( n \) is even, then the number of edges of \( C_n(t) \) is \( n + 1 \) which is odd. Thus, \( |e(0) - e(1)| = 1 \) in \( C_n(t) \). Without loss of generality, we assume that \( e(0) = e(1) + 1 \). If we remove the chord \((c_1, c_t)\) from \( C_n(t) \), then we get a new graph \( C_n \).

If the chord is labeled 0, then \( C_n \) has an edge-friendly labeling. From Theorem 2.2, the edge-balance index set of \( C_n \) is \( \{0\} \) since \( n \) is even. The three cases here are similar to the previous discussion. A similar argument shows that the edge-balance index set in this case is \( \{0, 1, 2\} \).

If the chord is labeled 1, then \( e_C(0) = e_C(1) + 2 \) in \( C_n \). By Lemma 2.1, we have \( v_C(0) = v_C(1) + 2 \). Again, a similar argument applies, and the edge-balance index set is \( \{0, 1, 2\} \).

Therefore, the edge-balance index set of \( C_n(t) \) is \( \{0, 1, 2\} \) if \( n \) is even. \( \square \)

**Example 5.** Figure 4 shows the edge-balance index sets of \( C_6(3) \) and \( C_6(4) \) are both \( \{0, 1, 2\} \).

![Figure 4: EBI of \( C_6(3) \) and \( C_6(4) \)](image)

**Example 6.** Figure 5 shows the balance index set of \( C_7(3) \) and \( C_7(4) \) are both \( \{0, 1\} \).

![Figure 5: Balance index of \( C_7(3) \) and \( C_7(4) \)](image)
3 Edge-balance index sets of dumbbell graphs

In this section, we consider the dumbbell graph $D(m, n)$ which is formed by joining two disconnected cycles $C_m$ and $C_n$ by an edge, where $m, n \geq 3$.

**Theorem 3.1** For $m, n \geq 3$, the edge-balance index set of the dumbbell graph is

$$\text{EBI}(D(m, n)) = \begin{cases} 
\{0, 1\} & \text{if } m + n \text{ is odd,} \\
\{0, 1, 2\} & \text{if } m + n \text{ is even.}
\end{cases}$$

**Proof.** A similar proof of Theorem 2.4 works here since the edge which connects two cycles plays the same role as a chord. \qed

**Example 7.** Figure 6 shows $\text{EBI}(D(3, 4))$ and $\text{EBI}(D(3, 6))$ are both $\{0, 1\}$ and $\text{EBI}(D(3, 3))$ and $\text{EBI}(D(3, 5))$ are both $\{0, 1, 2\}$.

4 Edge-balance index sets of $C(m, n)$

Let $G$ and $H$ be two graphs. Let $u$ be one of the vertices of $G$ and $v$ be one of the vertices of $H$. The one-point union of $(G, u)$ with $(H, v)$ is the graph...
obtained by identifying $u$ and $v$ from the disjoint union of $G$ and $H$. We use \( \text{Amal}(G, H, (u, v)) \) to denote the one-point union of $(G, u)$ and $(H, v)$.

For convenience, we denote \( \text{Amal}(C_m, C_n) \) by $C(m, n)$ and call it a **double cycle**. Note that, in order to make cycles, $m$ and $n$ must be greater or equal to 3.

We name the vertices of $C_m$ by $u_1, u_2, \cdots, u_m$ and the vertices of $C_n$ by $v_1, v_2, \cdots, v_n$. We also assume $C_m$ and $C_n$ amalgamate at $u_1$ and $v_1$ and call this amalgamated vertex $c$.

**Theorem 4.1** The edge-balance index set of the one-point join of two cycles $C(m, n)$ is

$$EBI(C(m, n)) = \begin{cases} 
\{0\} & \text{if } m = n = 3, \\
\{0, 1\} & \text{if } m + n \text{ is even and greater than 6,} \\
\{0, 1\} & \text{if } m + n = 7, \\
\{0, 1, 2\} & \text{if } m + n \text{ is odd and greater than 7.}
\end{cases}$$

**Proof.** For an edge-friendly labeling on $C(m, n)$, it gives the disjoint union of $C_m$ and $C_n$ an edge-friendly labeling since both graphs share the same set of edges. From Lemma 2.1, we know that the edge-balance index set of the disjoint union of $C_m$ and $C_n$ is $\{0\}$ if $m+n$ is even, or, $\{1\}$ if $m+n$ is odd. To obtain $C(m, n)$ from amalgamating $u_1$ and $v_1$ together, there are four possible cases:

1. If both $u_1$ and $v_1$ are labeled the same number, then after amalgamating, $c$ is labeled the same. Thus, the edge-balance index becomes 1 if $m + n > 6$ is even, or, 0 or 2 (depends on the number of the label of $c$ is greater or less the number of the other label) if $m + n > 7$ is odd.

2. If $u_1$ and $v_1$ are labeled differently, then after amalgamating, $c$ is not labeled because $u_1$ and $v_1$ provide the same numbers of 0- and 1-edges. Thus, the edge-balance index remains the same. Therefore, it is 0 if $m + n$ is even, or, 1 if $m + n$ is odd.

3. If $u_1$ is labeled and $v_1$ is not labeled, then after amalgamating, $c$ is labeled the same as $u_1$ since $v_1$ provides one 0-edge and one 1-edge. Thus, the edge-balance index remains the same. Therefore, it is 0 if $m + n$ is even, or, 1 if $m + n$ is odd.

4. If $u_1$ is not labeled and $v_1$ is labeled, then it is similar to the previous case. Thus, the edge-balance index is 0 if $m + n$ is even, or, 1 if $m + n$ is odd.
When \( m + n = 6 \), it is \( C(3, 3) \). It is impossible to have both \( u_1 \) and \( v_1 \) labeled 1 since it requires four 1-edges, and there are only three 1-edges in \( C(3, 3) \) for an edge-friendly labeling. Thus, \( \text{EBI}(C(3, 3)) = \{0\} \).

When \( m + n = 7 \), it is \( C(3, 4) \) or \( C(4, 3) \). In these two cases, since \( \text{EBI} = \{1\} \) for the disjoint union of \( C_3 \) and \( C_4 \), we have \( |v(0) - v(1)| = 1 \) in it. Without loss of generality, we assume that \( v(0) = v(1) + 1 \). Thus, by Lemma 2.1, we also have \( e(0) = e(1) + 1 \). When \( c \) is labeled 0, then we have one less vertex labeled 0 in \( C(3, 4) \) or \( C(4, 3) \). Thus, the edge-balance index is 0. It is impossible to have both \( u_1 \) and \( v_1 \) labeled 1 since it requires four 1-edges. It contradicts the facts \( e(0) + e(1) = 7 \) and \( e(0) = e(1) + 1 \). Thus, \( \text{EBI}(C(3, 4)) = \text{EBI}(C(4, 3)) = \{0, 1\} \).

Besides these two special cases, by the discussion of the above four cases, we conclude that the edge-balance index set \( \text{EBI}(C(m, n)) \) is \( \{0\} \) if \( m + n > 6 \) is even, or, \( \{0, 1, 2\} \) if \( m + n > 7 \) is odd.

\[ \text{Example 8.} \quad \text{Figure 7 shows } \text{EBI}(C(3, 4)) = \{0, 1\} \text{ and } \text{EBI}(C(4, 5)) = \{0, 1, 2\}. \]

\[ \text{Figure 7: EBI}(C(3, 4)) \text{ and EBI}(C(4, 5)) \]

\[ \text{Example 9.} \quad \text{Figure 8 shows } \text{EBI}(C(3, 3)) = \{0\} \text{ and } \text{EBI}(C(3, 5)) = \{0, 1\}. \]

\[ \text{Figure 8: EBI}(C(3, 3)) \text{ and EBI}(C(3, 5)) \]
5 Edge-Balance index sets of $\Phi_k$, $\Xi_k$, $\Theta_k$ and $\Psi_k$

In this section we investigate the edge-balance index sets of graphs in four families $\{\Phi_k, \Xi_k, \Theta_k, \Psi_k : k = 1, 2, \cdots \}$ of $(p, p+1)$-graphs which are the one-point union of path $P_{k+1}$ with a $(4,5)$-graph or a $(5,6)$-graph, respectively, displayed in Figure 9.

![Figure 9: Four families of $(p, p+1)$-graphs](image)

For convenience, we name each vertex as above throughout this section, and the vertex $v_0$ can also be called $u_0$.

When we remove a specific edge from a chosen vertex of order 3 and connect this edge to the vertex $u_k$, we transform $\Phi_k$ and $\Psi_k$ if $k \geq 2$ and $\Xi_k$ and $\Theta_k$ if $k \geq 1$ into $D(3, k+1)$, $D(4, k+1)$, $C(3, k+2)$ and $C(4, k+2)$, respectively, displayed in Figure 10.

**Lemma 5.1** Let $G$ be a graph of $\{\Phi_k, \Psi_k | k \geq 2 \} \cup \{\Xi_k, \Theta_k | k \geq 1 \}$ and $H$ be the transformed graph as above. Then the value of the edge-balance index of $G$ differs from the value of the edge-balance index of $H$ by at most one.

**Proof.** For an edge-friendly labeling of $G$, the transformation does not change the labels of the edges. It remains an edge-friendly labeling of $H$.

We call the chosen order 3 vertex $v$. We also name the edge of $v$ which is moved during the transformation by $e^m$ and the other two edges of $v$ by $e_1^x$ and $e_2^x$, respectively. Without loss of generality, we can assume that the edge $e^m$ is labeled 0 since we can swap 0 and 1 for every edge to get another edge-friendly labeling with the same edge-balance index.

The transformation only affects the labels of $u_k$ and $v$. Since we assume that $e^m$ is labeled 0, $u_k$ can only be labeled 0 or not labeled in $H$. Thus, we have the following six cases:

1. If both $u_k$ and $v$ are labeled 0 in $H$, then $(u_k, u_{k-1})$, $e_1^x$ and $e_2^x$ must be all labeled 0. Therefore, in $G$, both $u_k$ and $v$ remain labeled 0. Thus, the edge-balance index remains the same.
2. If \( u_k \) is labeled 0 and \( v \) is labeled 1 in \( H \), then \((u_k, u_{k-1})\) must be labeled 0 and \( e_1^x \) and \( e_2^x \) must be both labeled 1. Therefore, in \( G \), \( u_k \) remains labeled 0 and \( v \) remains labeled 1. Thus, the edge-balance index remains the same.

3. If \( u_k \) is labeled 0 and \( v \) is not labeled in \( H \), then \((u_k, u_{k-1})\) must be labeled 0 and \( e_1^x \) and \( e_2^x \) must be one labeled 0 and another one labeled 1. Therefore, in \( G \), \( u_k \) remains labeled 0 but \( v \) becomes labeled 0. Thus, we get one more 0-vertex.

4. If \( u_k \) is not labeled and \( v \) is labeled 0 in \( H \), then \((u_k, u_{k-1})\) must be labeled 1 and \( e_1^x \) and \( e_2^x \) must be both labeled 0. Therefore, in \( G \), \( u_k \) becomes labeled 1 but \( v \) remains labeled 0. Thus, we get one more 1-vertex.

5. If \( u_k \) is not labeled and \( v \) is labeled 1 in \( H \), then \((u_k, u_{k-1})\) must be labeled 1 and \( e_1^x \) and \( e_2^x \) must be both labeled 1. Therefore, in \( G \), \( u_k \) becomes labeled 1 but \( v \) remains labeled 1. Thus, we get one more 1-vertex.

6. If both \( u_k \) and \( v \) are not labeled in \( H \), then \((u_k, u_{k-1})\) must be labeled 1 and \( e_1^x \) and \( e_2^x \) must be one labeled 0 and another one labeled 1. Therefore, in \( G \), \( u_k \) becomes labeled 1 but \( v \) becomes labeled 0. Thus, the edge-balance index remains the same.
In all six cases, the edge-balance index is either unchanged or altered by 1. This completes the proof.

**Lemma 5.2** Let $G$ be any graph of $\{\Phi_k, \Psi_k | k \geq 2\} \cup \{\Xi_k, \Theta_k | k \geq 1\}$, and let $H$ be the graph in the same category with two more vertices, i.e., $\Phi_{k+2}$, $\Psi_{k+2}$, $\Xi_{k+2}$ or $\Theta_{k+2}$. Then, the edge-balance index set of $G$ is a subset of the edge-balance index set of $H$.

**Proof.** We add two vertices in the path part between $u_k$ and $u_{k-1}$ into $G$ to get $H$. In $H$, the old $u_k$ becomes $u_{k+2}$, and the old edge $(u_{k-1}, u_k)$ becomes $(u_{k+1}, u_{k+2})$. Thus, two new vertices are $u_k$ and $u_{k+1}$, and two new edges are $(u_{k-1}, u_k)$ and $(u_k, u_{k+1})$. If the edge $(u_{k+1}, u_{k+2})$ is labeled 0, then we can label the edge $(u_{k-1}, u_k)$ by 0 and the edge $(u_k, u_{k+1})$ by 1 without altering the edge-balance index. Similarly, if the edge $(u_{k+1}, u_{k+2})$ is labeled 1, then we can label the edge $(u_{k-1}, u_k)$ by 1 and the edge $(u_k, u_{k+1})$ by 0 without altering the edge-balance index. Since we add one 0-edge and one 1-edge, the labeling remains edge-friendly. This construction carries every edge-balance index value from $G$ to $H$. Therefore, the edge-balance index set of $G$ is a subset of the edge-balance index set of $H$. $\square$

**Theorem 5.3** For any $k \geq 1$, the edge-balance index set of $\Phi_k$ is

$$EBI(\Phi_k) = \begin{cases} \{0, 1, 2\} & \text{if } k \text{ is odd,} \\
0, 1, 2, 3 & \text{if } k \text{ is even.} \end{cases}$$

**Proof.** We first consider $\Phi_1$. For an edge-friendly labeling, $\Phi_1$ has three 0-edges and three 1-edges. To get an edge-balance index $|v(0) - v(1)| \geq 3$, it requires at least three 0-vertices. (Note that the roles of 0 and 1 are exchangeable. Thus, we focus on 0.) Since there are only three 0-edges, it is impossible to get four or more vertices labeled 0. In $\Phi_1$, it is easy to see that, no matter which three vertices are chosen to be labeled 0, there must be another one labeled 1. So, the edge-balance index can not exceed 2. Therefore, $EBI(\Phi_1)$ is a subset of $\{0, 1, 2\}$. We provides all three cases to confirm $EBI(\Phi_1) = \{0, 1, 2\}$.

For $k > 1$, by Lemma 5.1, the edge-balance index set of $\Phi_k$ is a subset of the edge-balance index set of $D(3, k + 1)$. By Theorem 3.1, $EBI(\Phi_k)$ is contained in $\{0, 1, 2\}$, if $k$ is odd, or, $\{0, 1, 2, 3\}$, if $k$ is even.

All four possible edge-balance indices $\{0, 1, 2, 3\}$ of $\Phi_2$ are provided...
to confirm \( \text{EBI}(\Phi_2) = \{0, 1, 2, 3\} \).

All three possible edge-balance indices \( \{0, 1, 2\} \) of \( \Phi_3 \) are provided

to confirm \( \text{EBI}(\Phi_3) = \{0, 1, 2\} \).

By Lemma 5.2, the result can be extended to all \( k \geq 4 \). This completes the proof. \( \square \)

**Example 10.** During the proof, we have shown that \( \text{EBI}(\Phi_k) = \{0, 1, 2\} \) for \( k = 1, 3 \) and \( \text{EBI}(\Phi_2) = \{0, 1, 2, 3\} \). Figure 11 shows \( \text{EBI}(\Phi_4) = \{0, 1, 2, 3\} \).

**Theorem 5.4** For any \( k \geq 0 \), The edge-balance index set of \( \Psi_k \) is

\[
\text{EBI}(\Psi_k) = \begin{cases} 
\{0\} & \text{if } k = 0, \\
\{0, 1, 2\} & \text{if } k = 1, \\
\{0, 1, 2\} & \text{if } k \geq 2 \text{ is even,} \\
\{0, 1, 2, 3\} & \text{if } k \geq 3 \text{ is odd.} 
\end{cases}
\]
Proof. We first consider $\Psi_0$. Since the graph $\Psi_0$ has six vertices, there must be three of them with label 0 and three of them with label 1.

Because of the symmetry, 0 and 1 can be swapped without altering the edge-balance index. Thus, we have only six possible edge-friendly labelings as follow:

Therefore, $\text{EBI}(\Theta_0) = \{0\}$.

Next, we consider $\Psi_1$. If we remove the edge $(u_0, u_1)$, we get a $\Psi_0$. For an edge-friendly labeling on $\Psi_1$, without loss of generality, we assume that there are 4 0-edges and 3 1-edges.

If $(u_0, u_1)$ is labeled 0, then the rest of the edge labeling is friendly on $\Psi_0$. Thus, by the above discussion, the edge-balance index is 0. Since $(u_0, u_1)$ is labeled 0, $u_1$ is labeled 0 in $\Psi_1$. The label of $u_0$ is either 0 or 1 in $\Psi_1$ which is determined by $v_0$ is unlabeled or labeled 1, respectively. Therefore, the edge-balance index is 0 or 2.

If $(u_0, u_1)$ is labeled 1, then there are 4 0-edges and 2 1-edges left to label $\Psi_0$. To get an edge-balance index more than 3, we need at least 4 0-vertices. According to the structure of $\Psi_0$, it is easy to see that there must be one vertex labeled 0 no matter which 4 edges to labeled 0. Therefore, the edge-balance index is less than or equal to 2.

All three possible edge-balance indices \(\{0, 1, 2\}\) of $\Psi_1$ are provided to confirm $\text{EBI}(\Psi_1) = \{0, 1, 2\}$.

For $k > 1$, by Lemma 5.1, the edge-balance index set of $\Psi_k$ is a subset of the edge-balance index set of $D(4, k + 1)$. By Theorem 3.1, $\text{EBI}(\Psi_k)$ is contained in \(\{0, 1, 2\}\), if $k > 1$ is even, or, \(\{0, 1, 2, 3\}\), if $k > 1$ is odd.

All three possible edge-balance indices \(\{0, 1, 2\}\) of $\Psi_2$ are provided.
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All four possible edge-balance indices \(\{0, 1, 2, 3\}\) of \(\Psi_3\) are provided to confirm \(\text{EBI}(\Psi_3) = \{0, 1, 2, 3\}\).

By Lemma 5.2, the result can be extended to all \(k \geq 4\). This completes the proof.

**Example 11.** During the proof, we have shown that \(\text{EBI}(\Psi_1) = \text{EBI}(\Psi_2) = \{0, 1, 2\}\) and \(\text{EBI}(\Psi_3) = \{0, 1, 2, 3\}\).

**Theorem 5.5** For any \(k \geq 1\), The edge-balance index set of \(\Xi_k\) is

\[
\text{EBI}(\Xi_k) = \begin{cases} 
\{0, 1\} & \text{if } k = 1, \\
\{0, 1, 2\} & \text{if } k = 2, \\
\{0, 1, 2\} & \text{if } k \text{ is odd and greater than } 1, \\
\{0, 1, 2, 3\} & \text{if } k \text{ is even and greater than } 2. 
\end{cases}
\]

**Proof.** By Lemma 5.1, the edge-balance index set of \(\Xi_k\) is a subset of the edge-balance index set of \(C(3, k + 2)\). By Theorem 4.1, \(\text{EBI}(\Xi_k)\) is contained in \(\{0, 1\}\) if \(k = 1\), or, \(\{0, 1, 2\}\) if \(k = 2\) or \(k > 1\) is odd, or, \(\{0, 1, 2, 3\}\) if \(k > 2\) is even.

All two possible edge-balance indices \(\{0, 1\}\) of \(\Xi_1\) are provided.
to confirm $\text{EBI}(\Xi_1) = \{0, 1\}$.

All three possible edge-balance indices $\{0, 1, 2\}$ of $\Xi_2$ or $\Xi_3$ are provided

\begin{align*}
\text{to confirm } & \text{EBI}(\Xi_2) = \text{EBI}(\Xi_3) = \{0, 1, 2\}. \\
\text{All four possible edge-balance indices } & \{0, 1, 2, 3\} \text{ of } \Xi_4 \text{ are provided}
\end{align*}

\begin{align*}
\text{to confirm } & \text{EBI}(\Xi_4) = \{0, 1, 2, 3\}. \\
\text{By Lemma 5.2, the result can be extended to all } k \geq 5. \text{ This completes } & \text{the proof.} \quad \square
\end{align*}

\begin{example}
During the proof, we have shown that $\text{EBI}(\Xi_1) = \{0, 1\}$, $\text{EBI}(\Xi_2) = \text{EBI}(\Xi_3) = \{0, 1, 2\}$ and $\text{EBI}(\Xi_4) = \{0, 1, 2, 3\}$.

\textbf{Theorem 5.6} For any $k \geq 0$, the edge-balance index set of $\Theta_k$ is

$$\text{EBI}(\Theta_k) = \begin{cases} 
\{0\} & \text{if } k = 0, \\
\{0, 1\} & \text{if } k = 2, \\
\{0, 1, 2\} & \text{if } k = 1 \text{ or } k = 3, \\
\{0, 1, 2\} & \text{if } k \text{ is even and greater than 2,} \\
\{0, 1, 2, 3\} & \text{if } k \text{ is odd and greater than 3}.
\end{cases}$$
\end{example}
Proof. Since \( \Theta_0 \) and \( \Psi_0 \) are the same graph, by Theorem 5.4, \( \text{EBI}(\Theta_0) = \{0\} \).

Assume \( k \geq 1 \). By Lemma 5.1, the edge-balance index set of \( \Theta_k \) is a subset of the edge-balance index set of \( C(4, k + 2) \). By Theorem 4.1, \( \text{EBI}(\Theta_k) \) is contained in \( \{0, 1, 2\} \) if \( k = 1 \) or \( k \) is even, or, \( \{0, 1, 2, 3\} \) if \( k > 1 \) is odd.

For \( k = 2 \), if we want to have an edge-balance index 2, we have to use the case 1 in the proof of Theorem 4.1, and start with a case of \( C(4, 4) \) with the edge-balance index 1. The case 1 requires all four edges incident to \( v_1 \) to be labeled by either 0 or 1. Without loss of generality, we assume that \( v_1 \) is labeled 1. Since there are only eight edges in \( C(4, 4) \), all other four edges must be labeled 0. So, the \( u_k \) and \( v_4 \) must be both labeled 0. This is the case 1 in the proof of Theorem 5.3. Thus, the edge-balance index remains the same. The cases 4, 5, 6 in the proof of Theorem 4.1 can only produce an edge-balance index 1. Therefore, it is impossible to have an edge-balance index 2 for \( \Theta_2 \).

Since \( \Theta_2 \) has only eight edges, two possible balance indices are \( \{0, 1\} \).

to confirm \( \text{EBI}(\Theta_2) = \{0, 1\} \).

For \( k = 3 \), if we want to have an edge-balance index 3, we have to use the case 1 in the proof of the Theorem 4.1, and start with a case of \( C(4, 5) \) with the edge-balance index 2. The case 1 requires all four edges incident to \( v_1 \) to be labeled by either 0 or 1 and the number of the label of \( v_1 \) is less then another label. Without loss of generality, we assume that \( v_1 \) is labeled 1. Since there are only eight edges in \( C(4, 5) \), all other five edges must be labeled 0. So, the \( u_k \) and \( v_4 \) must be both labeled 0. This is the case 1 in the proof of Theorem 5.3. Thus, the edge-balance index remains the same. The cases 4, 5, 6 in the proof of Theorem 4.1 can only produce an edge-balance index 1. Therefore, it is impossible to have an edge-balance index 3 for \( \Theta_3 \).

All three possible edge-balance indices \( \{0, 1, 2\} \) of \( \Theta_3 \) are provided

to confirm \( \text{EBI}(\Theta_3) = \{0, 1, 2\} \).

All three possible edge-balance indices \( \{0, 1, 2\} \) of \( \Theta_1 \) or \( \Theta_4 \) are provided
to confirm $\text{EBI}(\Theta_1) = \text{EBI}(\Theta_4) = \{0, 1, 2\}$.

All four possible edge-balance indices $\{0, 1, 2, 3\}$ of $\Theta_5$ are provided

to confirm $\text{EBI}(\Theta_5) = \{0, 1, 2, 3\}$.

By Lemma 5.2, the result can be extended to all $k \geq 6$. This completes the proof. \hfill $\square$

**Example 13.** During the proof, we have shown that $\text{EBI}(\Theta_2) = \{0, 1\}$, $\text{EBI}(\Theta_3) = \text{EBI}(\Theta_4) = \{0, 1, 2\}$ and $\text{EBI}(\Theta_5) = \{0, 1, 2, 3\}$.

Based on the above results, one may propose the following:

**Conjecture.** The numbers in $\text{EBI}(G)$ for any $(p, p+1)$-graph $G$ form an arithmetic progression.

**References**


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