Generalized Fibonacci-Like Polynomial and its
Determinantal Identities

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Abstract

It is well known that the Fibonacci polynomials are of great importance in the study of many subjects such as Algebra, geometry, combinatorics and number theory itself. Fibonacci polynomials defined by the recurrence relation \( f_n(x) = x f_{n-1}(x) + f_{n-2}(x), \ n \geq 2 \) with \( f_0(x) = 0, \ f_1(x) = 1 \). In this paper we introduce Generalized Fibonacci-Like Polynomials. Further we present its generalized determinantal identities with classical polynomials like Fibonacci Polynomial, Lucas Polynomials, Pell Polynomials and Pell-Lucas Polynomials.

Mathematics Subject Classification: 11B39, 11B37, 11C08, 11C20

Keywords: Fibonacci polynomial, Fibonacci-Like polynomial, Determinant

1. INTRODUCTION

Fibonacci polynomials defined by the recurrence relation \( f_n(x) = x f_{n-1}(x) + f_{n-2}(x), \ n \geq 2 \) with \( f_0(x) = 0, \ f_1(x) = 1 \). It is well known that
the Fibonacci polynomials are of great importance in the study of many subjects such as Algebra, geometry, combinatorics and number theory itself.

Many authors have studied Fibonacci polynomials and Generalized Fibonacci polynomials identities. They applied concept of Matrix and Determinants to establish some identities. Spivey [8] describe sum property for determinants and presented new proof identities like Cassini identity, d’Ocagne identity and Catalan identity. Köken and Bozkurt [6] define Jacobsthal M-matrix and Jacobsthal Q-matrix similar to Fibonacci Q-matrix and using these matrix representations to found the Binet like formula for jacobsthal numbers. A.J.Macfarlane [4] use the property for determinants and give new identities involving Fibonacci and related numbers. Some determinantal identities involving Fibonacci polynomials, Lucas polynomials, Chebyshev Polynomials, Pell polynomials, Pell-Lucas polynomials, Vieta-Lucas Polynomials are described [5]. In this paper, we introduce Generalized Fibonacci-Like Polynomials and its determinantal identities. Also we establish result in terms of Generalized Pell Polynomials and Generalized Pell-Lucas Polynomials.

2. GENERALIZED FIBONACCI-LIKE POLYNOMIALS

We define Generalized Fibonacci-Like Polynomials by recurrence relation,

\[ V_n(x) = xV_{n-1}(x) + V_{n-2}(x) \; ; \; n \geq 3 \quad with \quad V_1(x) = a, \; V_2(x) = bx \]  \hspace{1cm} [2.1]

First few polynomials are

\[ V_3(x) = bx^2 + a \]
\[ V_4(x) = (x^3 + x)b + ax \]
\[ V_5(x) = (x^4 + 2x^2)b + (x^2 + 1)a \]
\[ V_6(x) = (x^5 + 3x^3 + x)b + (x^3 + 2x)a \]
\[ \ldots \]

If \( a = b = 1 \), then
\[ f_n(x) = x f_{n-1}(x) + f_{n-2}(x) \; ; \; with \; f_1(x) = 1, \; f_2(x) = x \] (Fibonacci polynomials)

If \( a = 2, \; b = 1 \), then
\[ l_n(x) = x l_{n-1}(x) + l_{n-2}(x) \; ; \; with \; l_0(x) = 2, \; l_1(x) = x \] (Lucas polynomials)

If \( a = 1, \; b = 2 \), then
\[ P_n(x) = 2x P_{n-1}(x) + P_{n-2}(x) \; ; \; with \; P_1(x) = 1, \; P_2(x) = 2x \] (Pell polynomials)

If \( a = b = 2 \), then
\[ Q_{n-1}(x) = 2x Q_{n-2}(x) + Q_{n-3}(x) \; ; \; with \; Q_0(x) = 2, \; Q_1(x) = 2x \] (Pell-Lucas polynomials)
Now we define a family of Fibonacci-Like polynomial as

\[ V = \{ V_{n+i+1}^i(x), V_{n+i}^i(x), V_{n+i+m}^i(x), V_{n+i+m+j}^i(x), V_{n+m+j}^i(x) \}, \]

Where \( n, i, j, k, m \) are positive integers with \( 0 < k < i, i+1 < m, j = 1 \).

Then Generalized Fibonacci-Like polynomials are

\[
\begin{align*}
V_{n+i+j}^i(x) &= xV_{n+i}^i(x) + V_{n+k}^i(x) \quad [2.2] \\
V_{n+m+i}^i(x) &= xV_{n+i+m}^i(x) + V_{n+i}^i(x) \quad [2.3] \\
V_{n+m+i+j}^i(x) &= xV_{n+i+m+j}^i(x) + V_{n+i+m}^i(x) \quad [2.4]
\end{align*}
\]

If \((a,b) = (1,1)\), then \( V_{n+i+j}^i(x) = F_{n+i+j}^i(x) \), the Generalized Fibonacci Polynomials.

If \((a,b) = (2,1)\), then \( V_{n+i+j}^i(x) = L_{n+i+j}^i(x) \), the Generalized Lucas Polynomials.

If \((a,b) = (1,2)\), then \( V_{n+i+j}^i(x) = P_{n+i+j}^i(x) \), the Generalized Pell Polynomials.

If \((a,b) = (2,2)\), then \( V_{n+i+j}^i(x) = Q_{n+i+j}^i(x) \), the Generalized Pell-Lucas Polynomials.

If \((x,a,b) = (1,1,1)\), then \( V_{n+i+j}^i(l) = F_{n+i+j}^i \), the Generalized Fibonacci numbers.

If \((x,a,b) = (1,2,1)\), then \( V_{n+i+j}^i(l) = L_{n+i+j}^i \), the Generalized Lucas numbers.

If \((x,a,b) = (1,1,2)\), then \( V_{n+i+j}^i(l) = P_{n+i+j}^i \), the Generalized Pell numbers.

If \((x,a,b) = (1,2,2)\), then \( V_{n+i+j}^i(l) = Q_{n+i+j}^i \), the Generalized Pell-Lucas numbers.

### 3. DETERMINANTAL IDENTITIES

Now we present determinantal identities

**Theorem 1:** If \( n, i, j, k, m \) are positive integers with \( 0 < k < i, i+1 < m, j = 1 \), then

\[
\begin{vmatrix}
\frac{b^2V_{n+i}^2(x) + a^2V_{n+i}^2(x)}{V_{n+i,j}^i(x)} & V_{n+i,j}^i(x) & V_{n+i,j}^i(x) \\
\frac{bV_{n+k}^i(x)}{bV_{n+i}^i(x)} & \frac{a^2V_{n+i}^2(x) + V_{n+i,j}^i(x)}{bV_{n+i}^i(x)} & bV_{n+i}^i(x) \\
\frac{aV_{n+i}^i(x)}{aV_{n+i}^i(x)} & \frac{V_{n+i,j}^i(x) + b^2V_{n+i,j}^i(x)}{aV_{n+i}^i(x)}
\end{vmatrix} = 4abV_{n+i}^i(x)V_{n+i}^i(x)V_{n+i,j}^i(x)
\]

**Proof:** Let \( \Delta = \]

\[
\begin{vmatrix}
\frac{b^2V_{n+i}^2(x) + a^2V_{n+i}^2(x)}{V_{n+i,j}^i(x)} & V_{n+i,j}^i(x) & V_{n+i,j}^i(x) \\
\frac{bV_{n+k}^i(x)}{bV_{n+i}^i(x)} & \frac{a^2V_{n+i}^2(x) + V_{n+i,j}^i(x)}{bV_{n+i}^i(x)} & bV_{n+i}^i(x) \\
\frac{aV_{n+i}^i(x)}{aV_{n+i}^i(x)} & \frac{V_{n+i,j}^i(x) + b^2V_{n+i,j}^i(x)}{aV_{n+i}^i(x)}
\end{vmatrix}
\]

[3.1]
Assume $b V_{n+k}(x) = \alpha$, $a V_{ni}(x) = \beta$, then by [2.1] $V_{n+i+j}(x) = \alpha + x \beta$, Now

$$
\Delta = \begin{vmatrix}
\frac{\alpha^2 + \beta^2}{\alpha + x \beta} & \alpha + x \beta & \alpha + x \beta \\
\frac{\beta^2 + (\alpha + x \beta)^2}{\alpha} & \alpha & 0 \\
\frac{\beta^2 + (\alpha + x \beta)^2}{\beta} & 0 & (\alpha + x \beta)^2 \\
\end{vmatrix}
$$

Multiplying and divided $R_1$ by $(\alpha + x \beta), R_2$ by $\alpha, R_3$ by $\beta$ by

$$
\Delta = \frac{1}{\alpha \beta (\alpha + x \beta)} \begin{vmatrix}
\alpha^2 + \beta^2 & (\alpha + x \beta)^2 & (\alpha + x \beta)^2 \\
\alpha^2 & \beta^2 + (\alpha + x \beta)^2 & \alpha^2 \\
\beta^2 & 0 & \alpha^2 + (\alpha + x \beta)^2 \\
\end{vmatrix}
$$

Applying $R_1 \to R_1 + R_2 + R_3 \& R_2 - R_1 \to R_2 \& R_3 - R_1 \to R_3$

$$
\Delta = \frac{2}{\alpha \beta (\alpha + x \beta)} \begin{vmatrix}
\alpha^2 + \beta^2 & \beta^2 + (\alpha + x \beta)^2 & \alpha^2 + (\alpha + x \beta)^2 \\
\beta^2 & 0 & -(\alpha + x \beta)^2 \\
\alpha^2 & \alpha^2 + (\alpha + x \beta)^2 & 0 \\
\end{vmatrix}
$$

Expand along first row, we get

$$
\Delta = 4ab V_{ni}(x)V_{n+k}(x)V_{n+i+j}(x)
$$

Put $b V_{n+k}(x) = \alpha, a V_{ni}(x) = \beta, V_{n+i+j}(x) = \alpha + x \beta$

**Theorem 2:** If $n, i, j, k, m, r$ are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$
\begin{vmatrix}
\beta V_{mk}(x) & a V_{ml}(x)V_{n+i+j}(x) & b V_{mk}(x)V_{n+j}(x) + V_{ml}(x) \\
\beta V_{mk}(x) + b V_{ml}(x)V_{n+i+j}(x) & \beta V_{mk}(x)V_{n+i+j}(x) & b V_{mk}(x)V_{n+i+j}(x) + V_{ml}(x) \\
\beta V_{mk}(x) + b V_{ml}(x)V_{n+i+j}(x) & \beta V_{mk}(x)V_{n+i+j}(x) & b V_{mk}(x)V_{n+i+j}(x) + V_{ml}(x) \\
\end{vmatrix}
$$

**Proof:** Let $
=\begin{vmatrix}
\beta V_{mk}(x) & a V_{ml}(x)V_{n+i+j}(x) & b V_{mk}(x)V_{n+i+j}(x) + V_{ml}(x) \\
\beta V_{mk}(x) + b V_{ml}(x)V_{n+i+j}(x) & \beta V_{mk}(x)V_{n+i+j}(x) & b V_{mk}(x)V_{n+i+j}(x) + V_{ml}(x) \\
\beta V_{mk}(x) + b V_{ml}(x)V_{n+i+j}(x) & \beta V_{mk}(x)V_{n+i+j}(x) & b V_{mk}(x)V_{n+i+j}(x) + V_{ml}(x) \\
\end{vmatrix}
$

Assume $b V_{n+k}(x) = \alpha, a V_{ni}(x) = \beta$, then by [1.1] $V_{n+i+j}(x) = \alpha + x \beta$, Now

$$
\Delta = \begin{vmatrix}
\alpha^2 & \beta(\alpha + x \beta) & \alpha(\alpha + x \beta) + (\alpha + x \beta)^2 \\
\alpha^2 + \alpha \beta & \beta^2 & \alpha(\alpha + x \beta) \\
\alpha \beta & \beta^2 + \beta(\alpha + x \beta) & (\alpha + x \beta)^2 \\
\end{vmatrix}
$$

[3.2]
[3.3]
[3.4]
[3.5]
[3.6]
[3.7]
Taking $\alpha, \beta, (\alpha + x\beta)$ common from $C_1, C_2, C_3$ & Applying $R_2 \to R_2 - (R_1 + R_3)$

$$\Delta = \alpha \beta (\alpha + x\beta) \begin{vmatrix} \alpha & (\alpha + x\beta) \\ \beta & (\alpha + x\beta) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_3$ & Expansion by $R_2$

$$\Delta = (2\alpha \beta (\alpha + x\beta))^2$$

Put $bV_{nk}(x) = \alpha, aV_{ni}(x) = \beta, V_{n+1}(x) = \alpha + x\beta$, we get

$$\Delta = \left\{2abV_{ni}(x)V_{nk}(x)\right\}^2$$

**Theorem 3:** If $n, i, j, k, m$, are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$\begin{vmatrix} -bV_{nk}^2(x) & abV_{ni}(x)V_{nk}(x) & bV_{nk}(x)V_{n+i+j}(x) \\ abV_{ni}(x)V_{nk}(x) & -aV_{nk}^2(x) & aV_{ni}(x)V_{n+i+j}(x) \\ bV_{nk}(x)V_{n+i+j}(x) & aV_{ni}(x)V_{n+i+j}(x) & -V_{n+i+j}^2(x) \end{vmatrix} = \left\{2abV_{ni}(x)\right\}^2$$

**Theorem 4:** If $n, i, j, k, m$, are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$\begin{vmatrix} \alpha^2V_{ni}^2(x) + V_{ni}^2(x) & abV_{ni}(x)V_{nk}(x) & bV_{nk}(x)V_{n+i+j}(x) \\ abV_{ni}(x)V_{nk}(x) & bV_{nk}^2(x) & aV_{ni}(x)V_{n+i+j}(x) \\ bV_{nk}(x)V_{n+i+j}(x) & aV_{ni}(x)V_{n+i+j}(x) & -aV_{nk}^2(x) \end{vmatrix} = \left\{2abV_{ni}(x)\right\}^2$$

**Theorem 5:** If $n, i, j, k, m$, are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$\begin{vmatrix} 2V_{n+i+j}(x) + aV_{ni}(x) + bV_{nk}(x) & bV_{nk}(x) & aV_{ni}(x) \\ V_{n+i+j}(x) & 2bV_{nk}(x) + aV_{ni}(x) + V_{n+i+j}(x) & aV_{ni}(x) \\ V_{n+i+j}(x) & bV_{nk}(x) & 2aV_{ni}(x) + bV_{nk}(x) + V_{n+i+j}(x) \end{vmatrix} = 2\left\{aV_{ni}(x) + bV_{nk}(x) + V_{n+i+j}(x)\right\}^3$$

**Theorem 6:** If $n, i, j, k, m$, are positive integers with $0 < k < i, i+1 < m, j = 1$, then

$$\begin{vmatrix} 1 + bV_{nk}(x) & 1 & 1 \\ 1 + aV_{ni}(x) & 1 & 1 \\ 1 + 1 + V_{n+i+j}(x) & 1 + 1 + V_{n+i+j}(x) \end{vmatrix} = \left\{abV_{ni}(x)V_{nk}(x)V_{n+i+j}(x)\right\} \left\{\frac{1}{bV_{nk}(x)} + \frac{1}{aV_{ni}(x)} + \frac{1}{V_{n+i+j}(x)} + 1\right\}$$

Above Theorems 3 to 6 can be solved same as Theorem: 1.
4. CONCLUSION

This paper describes Generalized Fibonacci-Like polynomials and its determinantal identities. Also results derived in terms of classical polynomials like Fibonacci Polynomial, Lucas Polynomials, Pell Polynomials and Pell-Lucas Polynomials.

ACKNOWLEDGEMENT. The authors are grateful to the referees for their useful comments.

References


Received: January, 2012