Multiplicative Coupled Fibonacci Sequences

of Third Order

G. P. S. Rathore¹, Shweta Jain² and Omprakash Sikhwal³

¹Department of Mathematics, College of Horticulture, Mandsaur, India
gps_rathore20@yahoo.co.in

²Department of Mathematics, Mandsaur Institute of Technology, Mandsaur, India
shweta_maths@rediffmail.com

³Department of Mathematics, Mandsaur Institute of Technology, Mandsaur, India
opbhsikhwal@rediffmail.com

Abstract

Coupled Fibonacci sequences involve two sequences of integers in which the elements of one sequence are part of the generalization of the other and vice versa. K. T. Atanassov was first introduced coupled Fibonacci sequences of second order in additive form. In this paper, we present multiplicative coupled Fibonacci sequences of third order under two specific schemes.

Mathematics Subject Classification: 11B39, 11B37

Keywords: Fibonacci sequence, 2-Fibonacci sequence, multiplicative coupled Fibonacci sequence

1. Introduction

In the recent years coupled Fibonacci sequences are popularized. Much work has been done in this field but its multiplicative form is less known. The coupled Fibonacci sequence was first introduced by K. T. Atanassov [5] and also discussed many curious properties and new direction of generalization of Fibonacci sequence in [2], [3], [6] and [7]. He was defined and studied about four

Let \( \{\alpha_i\}_{i=0}^{\infty} \) and \( \{\beta_i\}_{i=0}^{\infty} \) be two infinite sequences and four arbitrary real numbers \( a, b, c, d \) be given. The four different multiplicative schemes for 2-Fibonacci sequences are as follows:

First scheme
\[
\alpha_{n+2} = \alpha_{n+1} \cdot \alpha_n, \quad n \geq 0
\]
\[
\beta_{n+2} = \beta_{n+1} \cdot \beta_n, \quad n \geq 0.
\]

Second scheme
\[
\alpha_{n+2} = \beta_{n+1} \cdot \alpha_n, \quad n \geq 0
\]
\[
\beta_{n+2} = \alpha_{n+1} \cdot \beta_n, \quad n \geq 0.
\]

Third scheme
\[
\alpha_{n+2} = \alpha_{n+1} \cdot \beta_n, \quad n \geq 0
\]
\[
\beta_{n+2} = \beta_{n+1} \cdot \alpha_n, \quad n \geq 0.
\]

Fourth scheme
\[
\alpha_{n+2} = \beta_{n+1} \cdot \beta_n, \quad n \geq 0
\]
\[
\beta_{n+2} = \alpha_{n+1} \cdot \alpha_n, \quad n \geq 0.
\]

B. Singh and O. Sikhwal [1] studied various results of second order. In this paper, we present some results on multiplicative coupled Fibonacci sequences of third order under two specific schemes.

2. Multiplicative Coupled Fibonacci Sequences of Third Order

Let \( \{\alpha_i\}_{i=0}^{\infty} \) and \( \{\beta_i\}_{i=0}^{\infty} \) be two infinite sequences and six arbitrary real numbers \( a, b, c, d, e, f \) be given. Multiplicative coupled Fibonacci sequences of third order are generated by the following eight different ways:

First scheme
\[
\alpha_{n+3} = \beta_{n+2} \cdot \beta_{n+1} \cdot \beta_n, \quad n \geq 0
\]
\[
\beta_{n+3} = \alpha_{n+2} \cdot \alpha_{n+1} \cdot \alpha_n, \quad n \geq 0.
\]

Second scheme
\[
\alpha_{n+3} = \beta_{n+2} \cdot \alpha_{n+1} \cdot \alpha_n, \quad n \geq 0
\]
\[
\beta_{n+3} = \beta_{n+2} \cdot \beta_{n+1} \cdot \beta_n, \quad n \geq 0.
\]

Third scheme
\[
\alpha_{n+3} = \alpha_{n+2} \cdot \alpha_{n+1} \cdot \beta_n, \quad n \geq 0
\]
\[
\beta_{n+3} = \alpha_{n+2} \cdot \beta_{n+1} \cdot \beta_n, \quad n \geq 0.
\]

Fourth scheme
\[
\alpha_{n+3} = \beta_{n+2} \cdot \beta_{n+1} \cdot \beta_n, \quad n \geq 0
\]
\[
\beta_{n+3} = \beta_{n+2} \cdot \alpha_{n+1} \cdot \beta_n, \quad n \geq 0.
\]

Fifth scheme
\[
\alpha_{n+3} = \alpha_{n+2} \cdot \alpha_{n+1} \cdot \alpha_n, \quad n \geq 0
\]
\[
\beta_{n+3} = \beta_{n+2} \cdot \beta_{n+1} \cdot \alpha_n, \quad n \geq 0.
\]
The few terms of schemes (2.1) & (2.2) are as below:

**First scheme (2.1)**

\[ a_0 = a, \quad b_0 = b; \quad n = 1, \quad a_1 = c, \quad b_1 = d; \quad n = 2, \quad a_2 = e, \quad b_2 = f; \quad n = 3, \quad a_3 = bdf, \quad b_3 = ace; \]
\[ n = 4, \quad a_4 = acdef, \quad b_4 = bcdef; \quad n = 5, \quad a_5 = abc^2de^2f^2, \quad b_5 = abcd^2e^3f^2. \]

**Second scheme (2.2)**

\[ a_0 = a, \quad b_0 = b; \quad n = 1, \quad a_1 = c, \quad b_1 = d; \quad n = 2, \quad a_2 = e, \quad b_2 = f; \quad n = 3, \quad a_3 = ace, \quad b_3 = bdf; \]
\[ n = 4, \quad a_4 = ac^2e^2, \quad b_4 = bd^2f^2; \quad n = 5, \quad a_5 = a^2c^3e^4, \quad b_5 = b^2d^3f^4. \]

### 3. Main Results

Now we present some results under schemes (2.1) & (2.2).

**First scheme (2.1)**

**Theorem (3.1):** For every integer \( n \geq 0 \):

(a) \( \beta_0, \quad a_{4n} = a_0, \quad \beta_{4n+4} = \beta_0, \quad \beta_{4n+4} = \beta_4, \quad \beta_{4n+5} = \beta_4, \quad \beta_{4n+6} = \beta_0, \quad \beta_{4n+6}. \)

(b) \( \beta_0, \quad a_{4n+5} = a_1, \quad \beta_{4n+5} = \beta_1, \quad \beta_{4n+5} = \beta_1, \quad \beta_{4n+5} = \beta_4, \quad \beta_{4n+6} = \beta_0, \quad \beta_{4n+6}. \)

(c) \( \beta_2, \quad a_{4n+6} = a_2, \quad \beta_{4n+6} = \beta_2, \quad \beta_{4n+6}. \)

**Proof:** We prove the above results by induction hypothesis.

(a) If \( n = 0 \) then \( \beta_0, \quad a_1 = a_0, \quad \beta_4 \)

\[ = \beta_0, \quad \beta_1, \quad \beta_1. \]

\[ = \beta_0, \quad \alpha_2, \quad \alpha_2, \quad \alpha_0, \quad \beta_2. \]

\[ = \beta_0, \quad \alpha_3, \quad \alpha_3, \quad \alpha_0, \quad \beta_3. \]

\[ = \alpha_0, \quad \beta_4. \]

Thus the result is true for \( n = 0 \).

Now assume that the result is true for some integer \( n \geq 1 \) then

\[ \beta_0, \quad a_{4n+8} = \beta_0, \quad \beta_{4n+7}, \quad \beta_{4n+6}, \quad \beta_{4n+5} \]

\[ = \beta_0, \quad a_{4n+6}, \quad a_{4n+5}, \quad a_{4n+4}, \quad a_{4n+3}, \quad a_{4n+2}, \quad a_{4n+1}, \quad a_{4n}, \quad a_{4n+5}. \]

\[ = \alpha_0, \quad \beta_{4n+4}, \quad \beta_{4n+5}, \quad \beta_{4n+6}, \quad a_{4n+5}. \]

\[ = \alpha_0, \quad a_{4n+7}, \quad a_{4n+6}, \quad a_{4n+5}. \]

\[ = \alpha_0, \quad a_{4n+8}. \]

Hence the result is true for all integers \( n \geq 0 \).
Similar proofs can be given for remaining parts (b) and (c).

**Theorem (3.2):** For every integer $n \geq 0$:

(a) $\alpha_{4n+4} = \frac{\prod_{i=0}^{4n+3} \alpha_i}{\prod_{i=0}^{4n} \beta_i}$   
(b) $\beta_{4n+4} = \frac{\prod_{i=0}^{4n+3} \beta_i}{\prod_{i=0}^{4n} \alpha_i}$

(c) $\alpha_{4n+5} = \frac{\beta_0}{\alpha_0} \cdot \frac{\prod_{i=0}^{4n+4} \alpha_i}{\prod_{i=0}^{4n+1} \beta_i}$   
(d) $\beta_{4n+5} = \frac{\alpha_0}{\beta_0} \cdot \frac{\prod_{i=0}^{4n+5} \beta_i}{\prod_{i=0}^{4n+1} \alpha_i}$

(e) $\alpha_{4n+6} = \frac{\beta_0 \cdot \beta_1}{\alpha_0 \cdot \alpha_1} \cdot \frac{\prod_{i=0}^{4n+5} \alpha_i}{\prod_{i=0}^{4n+2} \beta_i}$   
(f) $\beta_{4n+6} = \frac{\alpha_0 \cdot \alpha_1}{\beta_0 \cdot \beta_1} \cdot \frac{\prod_{i=0}^{4n+5} \beta_i}{\prod_{i=0}^{4n+2} \alpha_i}$

**Theorem (3.3):** For every integer $n \geq 0$:

$\alpha_{n+7} = \alpha_{n+5} \cdot \alpha_{n+4}^2 \cdot \alpha_{n+3}^3 \cdot \alpha_{n+2}^2 \cdot \alpha_{n+1}$

$\beta_{n+7} = \beta_{n+5} \cdot \beta_{n+4}^2 \cdot \beta_{n+3}^3 \cdot \beta_{n+2}^2 \cdot \beta_{n+1}$

Theorems (3.1), (3.2) & (3.3) can be proved by induction method.

Finally we present results for scheme (2.2).

**Second scheme (2.2)**

**Theorem (3.4):** For every integer $n \geq 0$:

(a) $\alpha_{4n+4} = \frac{\prod_{i=0}^{4n+3} \alpha_i}{\prod_{i=0}^{4n} \alpha_i}$   
(b) $\beta_{4n+4} = \frac{\prod_{i=0}^{4n+3} \beta_i}{\prod_{i=0}^{4n} \beta_i}$

(c) $\alpha_{4n+5} = \frac{\prod_{i=0}^{4n+4} \alpha_i}{\prod_{i=0}^{4n+1} \alpha_i}$   
(d) $\beta_{4n+5} = \frac{\prod_{i=0}^{4n+5} \beta_i}{\prod_{i=0}^{4n+1} \beta_i}$

(e) $\alpha_{4n+6} = \frac{\prod_{i=0}^{4n+5} \alpha_i}{\prod_{i=0}^{4n+2} \alpha_i}$   
(f) $\beta_{4n+6} = \frac{\prod_{i=0}^{4n+5} \beta_i}{\prod_{i=0}^{4n+2} \beta_i}$
Multiplicative coupled Fibonacci sequences

Theorem (3.5): For every integer $n \geq 0$:
\[
\alpha_{n+7} = \alpha_{n+5} \cdot \alpha_{n+4}^2 \cdot \alpha_{n+3}^3 \cdot \alpha_{n+2}^2 \cdot \alpha_{n+1},
\]
\[
\beta_{n+7} = \beta_{n+5} \cdot \beta_{n+4}^2 \cdot \beta_{n+3}^3 \cdot \beta_{n+2}^2 \cdot \beta_{n+1}.
\]
Theorems (3.4) & (3.5) can also be proved by induction method.

Conclusion

In this paper we described and extended Multiplicative coupled Fibonacci sequences of third order under two specific schemes. Similar results can be developed for other schemes.

Acknowledgement

We are thankful to anonymous referees for valuable suggestions.

REFERENCES


Received: February, 2012