Generalized Loss Variance Bounds

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Abstract

A structure of financial loss partition on a probability space is considered. To the events of a partition and to each financial loss it associates event based losses and gains. Event based loss variance bounds are obtained, which generalise previous inequalities by Kremer and the author. Sharpness and the maximum of the loss variance bounds under different constraints are also determined.

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1. Introduction

As a measure of maximum dispersion from the mean, upper bounds on variance have applications in all areas of theoretical and applied mathematical sciences (e.g. Seaman and Odell [14], Agarwal et al. [1], Barnett et al. [2]). Since the pricing and valuation of actuarial and financial risks often depends on variance, appropriate bounds are of considerable practical interest.

In previous work the author has considered a structure of financial loss on a probability space by representing the financial loss as difference between loss and gain (see [6], [7]). In this setting, the inequality of Bowers [4] on the mean loss, as well as inequalities by Kremer [12], Hürlimann [5] and Birkel [3] on the loss variance, are simple consequences of the non-negative property of variance functions. Sharpness and extremal properties of the variance bounds have been analysed following the original contribution by Schmitter [13]. These results have been applied in reinsurance mathematics to price stop-loss contracts (see [8]) and to show that the linear combination of proportional and stop-loss reinsurance is not optimal unless it is a pure stop-loss contract, at least in case the variance premium principle is used to set insurance prices ([9], proof of Theorem 2, [11]).

In the present follow-up, the introduced financial loss structure is extended to the financial loss partition structure alluded to in [7], Remarque 1.1. First, the
sample space of a probability space \((\Omega, A, P)\) is partitioned into disjoint events \(E_i, i \in S\), such that \(\Omega = \bigcup_{i \in S} E_i\). Then, to a financial loss \(X\), that is a measurable real-valued random variable on this probability space, one associates \(E_i\)-losses \(X_i = X \cdot I(E_i)\) and \(E_i\)-gains \(\bar{X}_i = -X \cdot I(\bar{E}_i)\), \(i \in S\), where \(I(\cdot)\) is the indicator function and \(\bar{E}_i\) is the complement event belonging to \(E_i\). For example, a simple partition into three events \(E_1 = \{X < \ell\}\), \(E_2 = \{\ell \leq X < h\}\), \(E_3 = \{X \geq h\}\), is able to model various regimes of low, normal and high losses according to which event is revealed when the financial loss is realised. The detailed construction is presented in Section 2. Generalised loss variance bounds for financial loss partitions are obtained in Section 3. Sharpness and maximum loss variance bounds under different constraints are derived in Section 4.

2. A financial loss partition structure

Let \((\Omega, A, P)\) be a probability space. We consider measurable real-valued random variables \(X\) on it, that is maps \(X: \Omega \to R\). Each \(X\) is interpreted as a financial loss such that for \(\omega \in \Omega\) the real number \(X(\omega)\) is the realization of a loss and profit function with \(X(\omega) \geq 0\) for a loss and \(X(\omega) < 0\) for a profit. It is assumed that the mean \(\mu\) and variance \(\sigma^2\) of \(X\) exist.

To each event \(E \in A\) one associates its complement event \(\bar{E} \in A\) such that \(P(\bar{E}) = 1 - P(E)\). For a finite subset \(S \subset N\) of indices, consider a partition of events \(E_i, i \in S\), such that \(\Omega = \bigcup_{i \in S} E_i\), \(E_i \cap E_j = \emptyset\) for all \(i \neq j\), which yields a structure of financial loss partition as follows. Let \(I: A \to \{0,1\}\) be the indicator function and define for \(i \in S\) the random variables

\[X_i = X \cdot I(E_i): \text{the amount to be paid if the event } E_i \text{ occurs, called } E_i\text{-loss}\]
\[\bar{X}_i = -X \cdot I(\bar{E}_i): \text{the amount gained if the event } \bar{E}_i \text{ occurs, called } E_i\text{-gain}\]

Since \(I(E_i) + I(\bar{E}_i) = 1\) one has \(X = X \cdot I(E_i) + X \cdot I(\bar{E}_i) = X_i - \bar{X}_i\), which as a difference between loss and gain justifies the interpretation of \(X\) as a financial loss. The negative value \(G = -X = \bar{X}_i - X_i\) is called financial gain.

**Examples 2.1**

(i) With \(S = \{1,2\}\) the partition \(E_1 = \{X > 0\}\), \(E_2 = \{X \leq 0\}\), yields the positive loss \(X_1 = X_+\) and the positive gain \(\bar{X}_1 = G_+\), which is the simplest financial loss structure studied in [6], [7].
(ii) With \( S = \{1, 2, 3\} \) a partition of the type \( E_1 = \{ X < \ell \} \), \( E_2 = \{ \ell \leq X < h \} \), \( E_3 = \{ X \geq h \} \), allows one to distinguish between a regime of “normal” losses revealed when the event \( E_2 \) occurs, and two regimes of “extremal” losses consisting of a regime of “low” losses revealed when \( E_1 \) occurs and a regime of “high” losses revealed when \( E_3 \) occurs.

(iii) It is also possible to specialise to insurance layered risks of degree \( m \geq 2 \), that is non-negative random variables \( X \) such that there exists a partition into disjoint events \( E_i, i = 1, \ldots, m \), of the sample space, which is of the type

\[
E_1 = \{ X \leq d_1 \}, \quad E_i = \{ d_{i-1} < X \leq d_i \}, \quad i = 2, \ldots, m-1, \quad E_m = \{ X \geq d_{m-1} \},
\]

with \( 0 < d_1 < d_2 < \ldots < d_{m-1} \). Since premiums for a given layer may vary considerably on the reinsurance market, it is of primordial importance for a cedent to know the extent of possible variation of the premiums in order to optimize his layer structure and the price to pay for reinsurance. We like to note that such kinds of chains of reinsurance layers often occur in practical work on (re)insurance captives and have been considered in [10].

For application in the actuarial and financial context, we will assume that the loss and gain events \( E_i, \bar{E}_i \) occur with non-zero probability, that is \( 0 < P(E_i), P(\bar{E}_i) < 1 \) for all \( i \in S \). In general, since \( E_i, \bar{E}_i \) are disjoint for \( i \in S \), and \( E_i, E_j \) are disjoint for \( i \neq j \), one has the following loss and gain identities of an arbitrary order \( n \in \mathbb{N} \):

\[
X_i^n + (-1)^n \cdot \bar{X}_i^n = X^n, \quad i \in S, \quad \sum_{i \in S} X_i^n = X^n. \quad (2.1)
\]

At the usual elementary level, one is only interested in first and second order moments, that is for each \( i \in S \) define

\[
M_i = E[X_i] \quad : \text{the mean } E_i\text{-loss}
\]

\[
M_{2,i} = E[X_i^2] \quad : \text{the mean squared } E_i\text{-loss}
\]

\[
V_i = M_{2,i} - M_i^2 = Var[X_i] \quad : \text{the } E_i\text{-loss variance}
\]

\[
\bar{M}_i = E[\bar{X}_i] \quad : \text{the mean } E_i\text{-gain}
\]

\[
\bar{M}_{2,i} = E[\bar{X}_i^2] \quad : \text{the mean squared } E_i\text{-gain}
\]

\[
\bar{V}_i = \bar{M}_{2,i} - \bar{M}_i^2 = Var[\bar{X}_i] \quad : \text{the } E_i\text{-gain variance}
\]
The identities (2.1) and (2.2) imply the following elementary relationships.

**Lemma 2.1.** A financial loss partition structure satisfies the “loss and gain parity” relations

\[ M_i - \overline{M}_i = \mu, \quad (2.3) \]
\[ M_{2,j} - \overline{M}_{2,j} = \mu^2 + \sigma^2, \quad (2.4) \]
\[ V_i + \overline{V}_i = \sigma^2 - 2 M_i \overline{M}_i, \quad (2.5) \]

for each \( i \in S \), and the identities

\[ \sum_{i \in S} M_i = \mu, \quad (2.6) \]
\[ \sum_{i \in S} M_{2,j} = \mu^2 + \sigma^2, \quad (2.7) \]
\[ \sum_{i \in S} V_i = \sigma^2 - \sum_{i \in S} M_i \overline{M}_i. \quad (2.8) \]

**Proof.** The relations (2.3)-(2.5) are (2.5)-(2.7) in [6]. The identities (2.6) and (2.7) follow immediately from (2.2) with \( n = 1,2 \). Using the latter relations one obtains

\[ \sum_{i \in S} V_i = \sum_{i \in S} M_{2,j} - \sum_{i \in S} M_i^2 = \mu^2 + \sigma^2 - \left( \sum_{i \in S} M_i \right)^2 + 2 \cdot \sum_{i < j} M_i M_j \]
\[ = \sigma^2 + 2 \cdot \sum_{i < j} M_i M_j. \quad (2.9) \]

From this one obtains for fixed \( k \in S \) using (2.6) and (2.3) that

\[ \sum_{i \in S} V_i = \sigma^2 + 2 M_k \cdot \sum_{i \neq k} M_i + 2 \cdot \sum_{i \neq j} M_i M_j = \sigma^2 + 2 M_k \cdot (\mu - M_k) + 2 \cdot \sum_{i \neq j} M_i M_j \]
\[ = \sigma^2 - 2 M_k \overline{M}_k + 2 \cdot \sum_{i \neq j} M_i M_j. \]

Summing over \( k \in S \) using (2.9) one gets (with \(|S|\) the cardinality of the set \( S \))

\[ |S| \cdot \sum_{i \in S} V_i = |S| \cdot \sigma^2 - 2 \cdot |S| \cdot M_k \overline{M}_k + (|S| - 2) \cdot 2 \cdot \sum_{i \neq j} M_i M_j \]
\[ = |S| \cdot \sigma^2 - 2 \cdot \sum_{k \in S} M_k \overline{M}_k + (|S| - 2) \cdot \left( \sum_{i \in S} V_i - \sigma^2 \right), \]
which implies immediately (2.7). \( \Diamond \)
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One observes that (2.5) expresses the second order total $E_i$-loss and $E_i$-gain variance in terms of the variance and the first order mean $E_i$-loss and $E_i$-gain. Similarly, the identity (2.8) decomposes the total loss variance, defined as the sum of all $E_i$-loss variances, in terms of the variance and all mean $E_i$-losses and $E_i$-gains. In special case $|S| = 2$ the relations (2.5) and (2.8) are equivalent.


The following variance inequalities generalize Theorems 2.1 and 2.2 in [6].

**Theorem 3.1.** If the loss probability $P(E_i), i \in S$, is unknown, then the $E_i$-loss variance satisfies the upper bound

$$V_i \leq \sigma^2 - \sum_{k \in S} M_k \overline{M}_k$$

(3.1)

**Proof.** Since variances are non-negative, this follows directly from (2.8). ◊

**Theorem 3.2.** If the loss probabilities $P(E_k), k \in S$, are known, then the $E_i$-loss variance satisfies the lower and upper bounds

$$\frac{1 - P(E_i)}{P(E_i)} \cdot M_i^2 \leq V_i \leq \sigma^2 - \sum_{k \in S} M_k \overline{M}_k - \sum_{k \neq i} \frac{1 - P(E_k)}{P(E_k)} \cdot M_k^2.$$  (3.2)

**Proof.** Conditioning on the event $E_i$, with $P(E_i) \neq 0$ by assumption, we have

$$Var[X|E_i] = E[X^2|E_i] - E[X|E_i]^2 = \frac{1}{P(E_i)} \cdot \left( M_{2,i} - M_i^2 \right) \geq 0,$$

which implies that

$$V_i = M_{2,i} - M_i^2 \geq \frac{M_i^2}{P(E_i)} - M_i^2 = \frac{1 - P(E_i)}{P(E_i)} \cdot M_i^2,$$

that is the lower bound in (3.2). The upper bound follows from the identity (2.8) using the lower bounds for $V_k, k \neq i$. ◊
4. Sharpness and maximum loss variance bounds.

It is natural to ask when the obtained loss variance bounds in Theorem 3.2 are sharp, that is attained for some financial loss partition structure. We show that this is the case for finite $|S|$-atomic financial losses, which are defined as follows. Let $x_i, i \in S,$ be the atoms of a standardised discrete random variable $X$ with mean zero and variance one and probabilities $p_i = P(E_i) \neq 0,1, i \in S.$ With an arbitrary mean $\mu$ and variance $\sigma^2$ a finite $|S|$-atomic financial loss satisfies the relationships $M_i = (\mu + \alpha_i) \cdot p_i, \ M_{2i} = (\mu + \alpha_i)^2 \cdot p_i, \ i \in S,$ hence also

$$V_i = \left(1 - \frac{p_i}{p_i}\right) \cdot M_i^2, \ i \in S.$$  \hspace{1cm} (4.1)

A comparison of (2.8) and (3.2) shows immediately that the equalities in (3.2) are attained for finite $|S|$-atomic financial losses, and one has for $i \in S$ the identity

$$V_i = \mu^2 + \sigma^2 - M_i^2 - \sum_{k \in S, k \neq i} \frac{1}{p_k} \cdot M_k^2.$$  \hspace{1cm} (4.2)

It is now possible to obtain maximum loss variance bounds in (3.2) by solving the following minimisation problem

$$\sum_{k \in S, k \neq i} \frac{1}{p_k} \cdot M_k^2 = \text{min},$$  \hspace{1cm} (4.3)

which may be constrained by some side condition.

In practical applications several alternative situations may be of interest. Since a detailed discussion of the special case $|S| = 2$ is found in [6], we assume here that $|S| \geq 3.$ For illustration, we analyse two main cases. First, we ask for the maximum $E_i$-loss variance bound over the space $D_{1i} = D(\mu, \sigma, p_i, M_i, k \in S)$ of all financial loss random variables with fixed mean $\mu$, variance $\sigma^2$, loss probability $p_i = P(E_i)$ and mean $E_k$-losses $M_k, k \in S.$ Then, we consider the maximum of $V_i$ over the space $D_{2i} = D(\mu, \sigma, M_i, p_k, k \in S)$ of all financial loss random variables with fixed mean $\mu$, variance $\sigma^2$, mean $E_i$-loss $M_i$, and loss probabilities $p_k = P(E_k), k \in S.$ Two distinguished results are obtained.
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**Theorem 4.1.** The maximum $E_i$-loss variance bound over the space of financial losses $D_{ij} = D(\mu, \sigma, p_i, M_k, k \in S)$, $i \in S$, $|S| \geq 3$, is given by

$$\max_{D_{ij}} \{V_i\} = \mu^2 + \sigma^2 + \left(\frac{p_i}{1 - p_i}\right) \cdot M_i^2 - \left(\frac{1}{1 - p_i}\right) \cdot \sum_{k \in S} M_k^2,$$  \hspace{1cm} (4.4)

and is attained at the $|S|$-atomic financial loss with atoms $\mu + \alpha_k = \frac{M_k}{p_k}$, $k \in S$, and probabilities

$$p_k = \frac{[M_k / M_r]}{\sum_{k \neq i} [M_k / M_r]} (1 - p_i), \quad k \neq i,$$  \hspace{1cm} (4.5)

where $M_r \neq 0$ for some $r \neq i$.

**Proof.** The stationary point of the Lagrange function

$$L(p_k, k \neq i, \lambda) = \sum_{k \neq i} \frac{1}{p_k} M_k^2 - \lambda \left(\sum_{k \neq i} p_k - (1 - p_i)\right)$$  \hspace{1cm} (4.6)

yields the relationship

$$\frac{M_s^2}{p_i} = \frac{M_r^2}{p_r}, \quad r \neq s, \quad r, s \neq i.$$  \hspace{1cm} (4.7)

With an index $r \neq i$ such that $M_r \neq 0$, one obtains $p_k = \frac{M_k}{M_r} \cdot p_r$, $k \neq i, r$.

Using the side condition $\sum_{k \neq i} p_k = 1 - p_i$ one gets (4.5). Inserting into the right-hand side of (4.2) the best upper bound (4.4) follows. \hfill \Box

**Theorem 4.2.** The maximum $E_i$-loss variance bound over the space of financial losses $D_{ij} = D(\mu, \sigma, M_i, p_k, k \in S)$, $i \in S$, $|S| \geq 3$, does not depend on $p_k$, $k \neq i$, and is given by

$$\max_{D_{ij}} \{V_i\} = \mu^2 + \sigma^2 - M_i \cdot \frac{(\mu - M_i)^2}{1 - p_i}.$$  \hspace{1cm} (4.8)
It is attained at the $|S|$-atomic financial loss with atoms $\mu + \sigma x_k = \frac{M_k}{p_k}$ and probabilities $p_k$, $k \in S$, where

$$M_k = \left( \frac{\mu - M_i}{1 - p_i} \right) \cdot p_k, \quad k \neq i.$$  \hspace{1cm} (4.9)

**Proof.** The stationary point of the Lagrange function

$$L(M_k, k \neq i, \lambda) = \sum_{k \neq i} \frac{1}{p_k} M_k^2 - \lambda \left( \sum_{k \neq i} M_k - (\mu - M_i) \right)$$  \hspace{1cm} (4.10)

yields the relationship

$$M_k = \frac{p_k}{p_r} M_r, \quad k \neq i, r,$$  \hspace{1cm} (4.11)

for some $r \neq i$. Using the side condition $\sum_{k \neq i} M_k = \mu - M_i$ one gets (4.9), which inserted into the right-hand side of (4.2) yields the best upper bound (4.8). \hfill \Diamond

**References**


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