Some Results on Same Membership Value in Any Fuzzy Subgroup of a Group

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Abstract
In this paper we choose a different approach to study about a fuzzy subgroup. We make our stress on membership values of the elements of a group in any fuzzy subgroup and come to know how it is a compulsion for different elements of G to have same membership values in any fuzzy subgroup.

Keywords: fuzzy sets; fuzzy subgroups, membership values

1. INTRODUCTION
Since Rosenfeld [1] gave the concept of fuzzy group, a lot of results on fuzzy groups appeared. W. M. Wu [10] introduced the concept of normal fuzzy subgroup. Mukherjee and Bhattacharya [4] gave the concept of fuzzy coset. Different authors studied the concept of fuzzy quotient group. By using the concept of level subgroups P. S. Das[6] proved that fuzzy subgroups of a finite group form a chain. Murali and Makamba in their papers[7],[8],[9] by using the concept of flag, chain, key chain, pinned flag, gave very important results on equivalence of fuzzy subgroups and counting the number of fuzzy subgroups of an abelian group. Sharma and Doda [5] gave the result on different possibilities of fuzzy subgroups of a cyclic group. But in this paper we tried to choose a different approach using membership values and gave some important results on same membership value of the elements of the group in fuzzy subgroup.
2. Preliminaries

**Definition 2.1: Fuzzy Set**: Let \( X \) be any set and \( \mu \) be a function with domain \( X \) and range \([0,1]\), that is, \( \mu : X \rightarrow [0,1] \), then \( \mu \) is called a fuzzy subset of \( X \) and \( \mu(x) \) is called membership value of \( x \) in \( \mu \).

**Definition 2.2: Fuzzy sub-group**: Let \( G \) be a group, then a fuzzy subset \( \mu \) of \( G \) is called fuzzy sub-group of \( G \) if

\[
\text{a)} \quad \mu(xy) \geq \min \{ \mu(x), \mu(y) \} \\
\text{b)} \quad \mu(x^{-1}) \geq \mu(x) \quad \forall x \in G
\]

**Theorem 2.3**: Let \( G \) be a group and \( \mu \) be any fuzzy subgroup of \( G \). If \( x, y \in G \) such that \( \mu(y) > \mu(x) \), then \( \mu(xy) = \mu(x) \).

3 RESULTS ON SAME MEMBERSHIP VALUES

**Theorem 3.1**: If \( x, y \in G \) such that \( \mu(y) > \mu(x) \) then
\[
\mu(xy^n) = \mu(xy^{n-1}) = \mu(y^{n-1}x) = \mu(y^n x) \quad \text{for all } n=1,2,3,4,\ldots
\]

**Proof**: Let \( \mu(y) > \mu(x) \).

Now \( \mu(xy) \geq \min \{ \mu(y), \mu(x) \} \)
\[
\Rightarrow \mu(xy) = \mu(x) \quad \text{as } \mu(y) > \mu(x) \quad (1.1)
\]

Similarly \( \mu(yx) = \mu(x) \) \quad (1.2)

Consider \( \mu(xy^2) \geq \min \{ \mu(xy), \mu(y) \} \)
\[
\Rightarrow \mu(xy^2) = \mu(xy) \quad \text{as } \mu(y) > \mu(xy) \quad (1.3)
\]

Similarly \( \mu(y^2x) = \mu(yx) \) \quad (1.4)

From (1.1), (1.2), (1.3), (1.4), we get
\[
\mu(y) > \mu(x) = \mu(xy) = \mu(yx) = \mu(xy^2) = \mu(y^2x) \quad (1.5)
\]

Again consider \( \mu(xy^3) \geq \min \{ \mu(xy^2), \mu(y) \} \)
\[
\Rightarrow \mu(xy^3) = \mu(xy^2) \quad \text{as } \mu(xy^2) < \mu(y) \quad (1.6)
\]

Similarly \( \mu(y^3x) = \mu(y^2x) \) \quad (1.7)

From (1.5), (1.6), (1.7), we get
\[
\mu(y) > \mu(x) = \mu(xy) = \mu(yx) = \mu(xy^2) = \mu(y^2x) = \mu(xy^3) = \mu(y^3x)
\]

Continuing in this way, we get
\[
\mu(y) > \mu(xy^n) = \mu(xy^{n-1}) = \mu(y^{n-1}x) = \mu(y^n x) \quad \text{for all } n=1,2,3,4,\ldots
\]

**Corollary 3.2**: Let \( x, y \in G \) such that \( \mu(y) > \mu(x) \) and let \( H \) be a cyclic subgroup of \( G \) generated by \( y \), then the membership value of each element of left coset \( xH \) of \( H \) is equal and also equal to the membership value of each element of right coset \( Hx \) of \( H \) and equal to the membership value of \( x \) that is

\[ (a) \quad \mu(xh) = \mu(hx) \quad \forall h \in H \]
Some results on same membership value

(b) \( \mu(xh) = \mu(x) \quad \forall h \in H \)

**Theorem 3.3:** Let \( G \) be a group and let \( x \in G \) be such that \( o(a) \geq 2 \). Then if the membership values of \( a \) and \( a^2 \) in any fuzzy subgroup \( \mu \) of \( G \) are not equal then the membership values of all odd powers of \( a \) in \( \mu \) are equal, that is, if \( \mu(a) \neq \mu(a^2) \)

Then \( \mu(a^{2n-1}) = \mu(a^{2n+1}) \) for \( n = 1, 2, 3, \ldots \)

**Proof:** We know that \( \mu(a^2) \geq \mu(a) \) always

But \( \mu(a^2) \neq \mu(a) \Rightarrow \mu(a^3) > \mu(a) \)

Then by using theorem 3.1, we get

\[ \mu(a.a^{2n}) = \mu(a.a^{2n-2}) \text{ for } n = 1, 2, 3 \ldots \]

\[ \Rightarrow \mu(a^{2n+1}) = \mu(a^{2n-1}) \text{ for } n = 1, 2, 3 \ldots \]

This proves the result.

**Theorem 3.4:** If \( G \) is any finite cyclic group of order \( n \) and \( \mu \) be any fuzzy subgroup of \( G \), then at least \( \frac{n}{2} \) elements of \( G \) have the least membership value in \( \mu \).

**Proof:** Let \( G \) be a finite cyclic group of order \( n \) generated by \( a \).

Case (i) If \( \mu(a) \neq \mu(a^2) \), then by theorem 3.3, we see that images of all odd powers of \( a \) are equal, that is, at least \( \frac{n}{2} \) elements of \( G \) have same membership value in \( \mu \).

Also we know that the membership value of generator of \( G \) is least in any fuzzy subgroup \( \mu \) of \( G \). So we can say that at least \( \frac{n}{2} \) elements of \( G \) have the least membership value in \( \mu \).

Case (ii) If \( \mu(a) = \mu(a^2) \) and let \( \mu(a) = \mu(a^2) \neq \mu(a^3) \), then

\[ \mu(a^3) > \{ \mu(a) = \mu(a^2) \} \text{, then by theorem 3.1, we get} \]

\[ \{ \mu(a) = \mu(a^2) = \mu(a^3) = \ldots \} = \{ \mu(a^2) = \mu(a^3) = \mu(a^5) = \ldots \} \]

That is, more than \( \frac{n}{2} \) elements of \( G \) have the least membership value in \( \mu \).

Continuing in this way, in each case, the elements with least membership value will increase.

**Theorem 3.5:** If \( G \) is any finite cyclic group of odd order and \( \mu \) be any fuzzy sub-group of \( G \), then at least \( \frac{2r}{3} \) elements of \( G \) have the least membership value in \( \mu \).

**Proof:** Let \( G = \langle a \rangle \) be a cyclic group with odd order \( n \), then

\( \mu(a) = \mu(a^2) \) as both \( a \) and \( a^2 \) are generators of \( G \).

Case (i) Let \( \mu(a) = \mu(a^2) \neq \mu(a^3) \)
That is, \( \mu(a^3) > \mu(a) = \mu(a^2) \), then by theorem 3.1, we get
\[
\{ \mu(a) = \mu(a^3) = \mu(a^5) = \ldots \} = \{ \mu(a^2) = \mu(a^5) = \mu(a^8) = \ldots \}
\]
\( \Rightarrow \) at least \( \frac{2n}{3} \) elements of \( G \) have least membership value in \( \mu \)

Case (ii) If \( \mu(a) = \mu(a^3) = \mu(a^5) \neq \mu(a^4) \)
\( \Rightarrow \mu(a^4) > \mu(a) = \mu(a^2) = \mu(a^3) \)

Then by theorem 3.1, we get
\[
\{ \mu(a) = \mu(a^5) = \mu(a^9) = \ldots \} = \{ \mu(a^2) = \mu(a^6) = \mu(a^{10}) = \ldots \}
\]
\( = \{ \mu(a^3) = \mu(a^7) = \mu(a^{11}) = \ldots \} \)

Which shows that more than \( \frac{2n}{3} \) elements of \( G \) have least membership value in \( \mu \)

Continuing in this way, in each case, the elements with least membership value will increase.

**Theorem 3.6:** Let \( G \) be any group and \( x \in G \) be any element of order \( n \), then for any integer \( m \) such that \( (m, n) = 1 \), then the membership value of \( x^m \) and \( x \) in any fuzzy subgroup \( \mu \) of \( G \) are equal.

**Proof:** As we know for any integer \( m \), \( \mu(x^m) \geq \mu(x) \) \hspace{1cm} (1.8)

Now \( (m, n) = 1 \) \( \Rightarrow \exists \) integers \( k_1 \) and \( k_2 \) such that \( mk_1 + nk_2 = 1 \)
\( \Rightarrow x^{mk_1 + nk_2} = x \)
\( \Rightarrow x^{mk_1} (x^n)^{k_2} = x \)
\( \Rightarrow x^{mk_1} = x \) as \( o(x) = n \)

Now \( \mu(x^{mk_1}) \geq \mu(x^m) \)
\( \Rightarrow \mu(x) \geq \mu(x^m) \) \hspace{1cm} (1.9)

From (1.8) and (1.9), we get
\( \mu(x) = \mu(x^m) \)

**Corollary 3.7:** For any element \( x \) of \( G \) of order \( n \), there are at least \( \phi(n) \) elements in \( G \) whose membership values in any fuzzy subgroup \( \mu \) of \( G \) are equal to \( \mu(x) \).

**Corollary 3.8:** If \( O(x) = p \), where \( p \) is prime, then the membership values of \( x, x^2, x^3, \ldots, x^{p-1} \) in any fuzzy subgroup \( \mu \) of \( G \) are equal.

**Corollary 3.9:** Membership values of \( x^p \) and \( x^q \) in any fuzzy subgroup \( \mu \) of \( G \) are equal if they are of same order.
4 DIFFERENT POSSIBILITIES OF MEMBERSHIP VALUES OF A DIHEDRAL GROUP $G = D_{2n}$, (WHERE $n$ IS A PRIME NUMBER) IN ANY FUZZY SUBGROUP OF $D_{2n}$

Let $G = D_{2n} = \{ x^i y^j \mid i = 0, 1, j = 0, 1, 2, \ldots, n-1 \text{ where } x^2 = e = y^n, xy = y^{-1}x \}$ be a Dihedral group (where $n$ is a prime number) then for any fuzzy subgroup $\mu$ of $G$, the different possibilities of all elements of $G$ in $\mu$ are as follows:

**Case 1:** If $\mu(y) > \mu(x)$ as $<y>$ is a cyclic group of prime order $n$ so the membership values of $y, y^2, \ldots, y^{n-1}$ in $\mu$ are equal. Also by using theorem 3.1, the membership values of all the element of $G$ in $\mu$ are given by

$$\mu(e) \geq \{ \mu(y) = \mu(y^2) = \mu(y^3) = \ldots = \mu(y^{n-1}) \} > \{ \mu(x) = \mu(xy) = \ldots = \mu(xy^{n-1}) \}$$

**Case 2:** If $\mu(x) > \mu(y)$, then by theorem 3.1

$$\mu(e) \geq \mu(x) > \mu(y) = \mu(y^2) = \mu(y^3) = \ldots = \mu(y^{n-1}) = \mu(x) = \mu(xy) = \ldots = \mu(xy^{n-1})$$

**Case 3:** If $\mu(x) = \mu(y)$ Then

$$\mu(e) \geq \mu(x) \geq \mu(xy^k) \geq \{ \mu(y) = \mu(y^2) = \mu(y^3) = \ldots = \mu(y^{k-1}) = \mu(y^{k+1}) = \ldots = \mu(y^{n-1}) \}$$

Where $k = 1, 2, 3, \ldots, n-1$

**REFERENCES**