Fractional Calculus of a k-Wright Type Function

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Abstract
This article deals with a function \( W_{\gamma,k,\alpha,\beta}(z) \) that in the case of \( \gamma = 0 \) and \( k = 1 \) reduces to the classical Wright function \( W_{\alpha,\beta}(z) \).

Some elementary properties of the new \( W_{\gamma,k,\alpha,\beta}(z) \) are presented and its Laplace transform is obtained.

Also it has been shown that the fractional Riemann-Liouville integral transform such functions with powers multipliers into functions of the same form with a very interesting relation between indices.

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I Introduction and Preliminaries

The importance of the role played by the Wright function in partial differential equation of fractional order is well known and was widely treated in papers by several authors including Gorenflo, Luchko, Mainardi (cf. [5]), Mainardi (cf. [7]), Mainardi, Pagnini (cf. [8]).

Since Diaz and Pariguan (cf. [2]) introduced be the k-Pochhammer symbol and the k-Gamma function as a continuous deformation of the classic Euler Gamma function, many jobs issue dedicated to what you could appoint k-calculus have been published. Among them we highlight [2], [4], [9], [10].

In this article we introduce a generalization of the Wright function that we will named the k-Wright function, in whose definition is used the k-Gamma function \( \Gamma_k(z) \) and the k-Pochhammer symbol \( (\gamma)_{n,k} \) and we will denoted by \( W_{\gamma,k,\alpha,\beta}(z) \).

In Section 1 we present the elementary and essential definitions concerning to the fractional calculus which are necessary along the paper and introduce the series representation of the generalized k-Wright function \( W_{\gamma,k,\alpha,\beta}(z) \).
In Section 2 we consider some elementary properties. Also we obtain it Laplace Transform whose expression contains the k-Mittag-Leffler function $E_{k,\alpha,\beta}(z)$.

Finally in Section 3 we consider the auxiliary function $\mathcal{E}(t,k,\alpha,\beta)$ and evaluate it Riemann-Liouville fractional integral and also it Riemann-Liouville derivative.

In the development of this paper we use fractional integrals and fractional derivatives, and also Laplace transform, so we introduce the definitions and notations.

**Definition 1** Let $f$ be a sufficiently well-behaved function with support in $\mathbb{R}^+$, and let $\nu$ be a real number, $\nu > 0$. The Riemann-Liouville fractional integral of order $\nu$, $I_{+}^{\nu}f$ is given by

$$I_{+}^{\nu}f(t) = \frac{1}{\Gamma(\nu)} \int_{0}^{t} (t-\tau)^{\nu-1} f(\tau)d\tau$$

where $\Gamma(z)$ is the Euler Gamma function

$$\Gamma(z) = \int_{0}^{\infty} e^{-t}t^{z-1}dt$$

for $\text{Re}(z) > 0$, $z$ a complex number.

It is known that the semigroup property is verified

$$I_{+}^{\nu}I_{+}^{\eta} = I_{+}^{\nu+\eta}$$

where by $I_{+}^{0}$ we denote the Identity operator (cf. [12]), (cf. [3]), (cf. [11]).

The Riemann-Liouville fractional derivative of order $\nu > 0$, $D_{+}^{\nu}$ is defined as the left inverse of the Riemann-Liouville integral of order $\nu$; i. e,

$$D_{+}^{\nu}I_{+}^{\nu} = I, \quad \nu > 0 \quad \text{cf. [11]}$$

Other way of defined this fractional derivative is the follow.

**Definition 2** Let $\nu$ be a real number, and let $m$ be the integer sucht that $m - 1 < \nu \leq m$. Then the Riemann-Liouville fractional derivative of order $\nu$ is given by

$$D_{+}^{\nu}f(t) = D^{m}I_{+}^{m-\nu}f(t)$$
Equivalently, we have

\[ D^\nu f(t) = \begin{cases} \frac{d^m}{dt^m} \left[ \frac{1}{\Gamma(m-\nu)} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{m-\nu}} \right], & m - 1 < \nu \leq m \\ \frac{d^m}{dt^m} f(t), & \nu = m \end{cases} \]  

Let \( \mathcal{L}(f)(z) \) be the Laplace transform of an exponential order function and piecewise continuous where

\[ \mathcal{L}(f)(z) = \int_0^\infty e^{-zt} f(t)dt \]  

\( t \in \mathbb{R}^+, \) and \( z \in \mathbb{C}. \)

For further development we point out the action of the Laplace transform on Riemann-Liouville fractional integral and respect to the Riemann-Liouville fractional derivative.

For the Riemann-Liouville fractional integral we have

\[ \mathcal{L}\left[I^\nu f\right](z) = \frac{\mathcal{L}(f)(z)}{z^\nu} \]  

and for the Riemann-Liouville fractional derivative

\[ \mathcal{L}\left[D^\nu f\right](z) = z^\nu \mathcal{L}(f)(z) - \sum_{k=0}^{m-1} \left[ D^k I^{m-k}_+ f(0^+) \right] z^{m-k-1}; \quad m - 1 < k \leq m \]  

cf.[11], cf.[3].

Since Diaz y Pariguan (cf. [1]) introduce the k-Gamma function \( \Gamma_k(z) \) as

\[ \Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t}{k}} dt \]  

generalization of the so called special functions as the Beta, the Zeta, and the Mittag-Leffler functions have their k-extensions, and also a k-fractional integral (cf. [Mubeen]) is defined.

In a recent paper, Dorrego and Cerutti (cf. [2]) have introduced a k-generalization of the classical Mittag-Leffler function \( E_{k,\alpha,\beta}^\xi(z) \) of the form

\[ E_{k,\alpha,\beta}^\xi(z) = \sum_{n=0}^{\infty} \frac{\xi_{n,k}}{\Gamma_k(n\alpha + \beta)} \frac{z^n}{n!} \]  

(I.10)
where \( k \in \mathbb{R} \), \( \alpha, \beta \) and \( \xi \) are complex numbers that \( \text{Re}(\alpha) > 0 \), \( \text{Re}(\beta) > 0 \), and \((\xi)_{n,k}\) denote the Pochhammer k-symbol defined by (cf. [1]) \((\xi)_{n,k} = \xi(\xi+k)(\xi+2k)\cdots(\xi+(n-1)k)\); \( n = 1, 2, \ldots \).

Easily it can be proved that when \( \xi = 1 \), and \( k = 1 \) (I.10) reduce to classical two parameters Mittag-Leffler function.

It is known that the Wright function \( W_{\alpha,\beta} \) is defined by the series

\[
W_{\alpha,\beta} = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma \left(\alpha n + \beta\right)}; \quad \alpha > -1; \quad \beta \in \mathbb{C}
\]  

where \( \Gamma(z) \) is the Euler Gamma function.

By means of the series development a generalization of (I.12) is now introduced we put the following

**Definition 3** Let \( k \in \mathbb{R} \); \( \alpha, \beta, \gamma \in \mathbb{C} \), \( \text{Re}(\alpha) > 0 \), \( \text{Re}(\beta) > 0 \). The \( k \)-Wright function is defined as

\[
W_{\gamma k,\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)(n!)^2} z^n
\]  

where \((\gamma)_{n,k}\) is the \( k \)-Pochhammer symbol given by \((\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)\cdots(\gamma+(n-1)k); \ n = 1, 2, \ldots, \) and \( \Gamma_k(z) \) is the \( k \)-Gamma function given by (I.9).

Easily we can prove that \( W_{\gamma k,\alpha,\beta}(z) \rightarrow W_{\alpha,\beta}(z) \) as \( k \rightarrow 1 \) and \( \gamma = 1 \), because \( \Gamma_k(z) \rightarrow \Gamma(z) \) and \((\gamma)_{n,k} \rightarrow (\gamma)_n\).

## II Main Results

### II.1 Elementary Properties of the \( k \)-Wright function

In this section we obtain several elementary properties of our \( k \)-Wright function \( W_{\gamma k,\alpha,\beta}(z) \) defined by (I.12) and some others associated with the \( k \)-Mittag-Leffler function obtained by means of the Laplace Transform.

**Lemma 1** Let \( \alpha, \beta, \gamma \in \mathbb{C} \), \( \text{Re}(\alpha) > 0 \), \( \text{Re}(\beta) > 0 \). Then there holds

\[
W_{\gamma k,\alpha,\beta}(z) = \beta W_{\gamma,\alpha,\beta+k}(z) + \alpha z \frac{d}{dz} W_{\gamma,\alpha,\beta+k}(z)
\]  

(II.1)
Proof. By definition (I.3) we have

\[ \beta W_{k,\alpha,\beta+k}^\gamma(z) + \alpha z \frac{d}{dz} W_{k,\alpha,\beta+k}^\gamma(z) = \]

\[ \frac{\beta}{\alpha} \sum_{n=0}^{\infty} (\gamma)_n \frac{\Gamma_{k}(\alpha n + \beta + k)(n!)^2}{\Gamma_{k}(\alpha n + \beta + k)(n!)^2} z^n = \]

\[ \sum_{n=0}^{\infty} \frac{\beta(\gamma)_n \Gamma_{k}(\alpha n + \beta + k)(n!)^2}{\Gamma_{k}(\alpha n + \beta + k)(n!)^2} z^n \]

Taking into account \( \Gamma_{k}(z + k) = z \Gamma_{k}(z) \), (II.2) it result

\[ \beta W_{k,\alpha,\beta+k}^\gamma(z) + \alpha z \frac{d}{dz} W_{k,\alpha,\beta+k}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\alpha n + \beta)(\gamma)_n \Gamma_{k}(\alpha n + \beta)(n!)^2}{\Gamma_{k}(\alpha n + \beta)(n!)^2} z^n = \]

\[ \sum_{n=0}^{\infty} \frac{(\alpha n + \beta)(\gamma)_n \Gamma_{k}(\alpha n + \beta)(n!)^2}{\Gamma_{k}(\alpha n + \beta)(n!)^2} z^n \]

Then, we have

\[ W_{k,\alpha,\beta+k}^\gamma(z) = \beta W_{k,\alpha,\beta+k}^\gamma(z) + \alpha z \frac{d}{dz} W_{k,\alpha,\beta+k}^\gamma(z) \]

Lemma 2 Let \( \alpha, \beta, \gamma \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0 \). Then there holds the formula

\[ \frac{d}{dz} W_{k,\alpha,\beta}^\gamma(z) = \gamma \sum_{n=0}^{\infty} \frac{(\gamma + k)_n \Gamma_{k}(\alpha n + \alpha + \beta)(n+1)(n!)^2}{\Gamma_{k}(\alpha n + \alpha + \beta)(n+1)(n!)^2} z^n \]

Proof. From definition (I.3) and using the well known relations for the k-Pochhammer symbol \( (\gamma)_{n+1,k} = \gamma(\gamma + k)_{n,k} \) we have

\[ \frac{d}{dz} W_{k,\alpha,\beta}^\gamma(z) = \frac{d}{dz} \sum_{n=0}^{\infty} \frac{(\gamma)_n \Gamma_{k}(\alpha n + \beta)(n!)^2}{\Gamma_{k}(\alpha n + \beta)(n!)^2} z^n = \]

\[ \sum_{n=1}^{\infty} \frac{(\gamma)_n \Gamma_{k}(\alpha n + \beta)(n!)^2}{\Gamma_{k}(\alpha n + \beta)(n!)^2} n z^{n-1} = \sum_{n=0}^{\infty} \frac{(\gamma)_{n+1,k} \Gamma_{k}(\alpha(n+1) + \beta)(n+1)(n!)^2}{\Gamma_{k}(\alpha(n+1) + \beta)(n+1)(n!)^2} z^n = \]
\[ \gamma \sum_{n=0}^{\infty} \frac{(\gamma + k)_{n,k}}{\Gamma_k(\alpha n + \alpha + \beta) \ (n+1)(n!)^2} z^n \]

Furthermore, it can be proved that \( W_{k,\alpha,\beta}^{\gamma+k}(z) \) verified the following differential equation

\[ W_{k,\alpha,\beta}^{\gamma+k}(z) = \left( \frac{k}{\gamma} \right) z \frac{d}{dz} W_{k,\alpha,\beta}^{\gamma}(z) + W_{k,\alpha,\beta}^{\gamma}(z) \]

**Lemma 3** Let \( \alpha, \beta, \gamma \in \mathbb{C} \), \( \text{Re}(\alpha) > 0 \), \( \text{Re}(\beta) > 0 \). Then

\[ W_{k,\alpha,\beta}^{\gamma+k}(z) - W_{k,\alpha,\beta}^{\gamma}(z) = \left( \frac{k}{\gamma} \right) z \frac{d}{dz} W_{k,\alpha,\beta}^{\gamma}(z) \quad (\text{II.4}) \]

**Proof.** From (I.3) we have

\[ W_{k,\alpha,\beta}^{\gamma+k}(z) - W_{k,\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma + k)_{n,k} \ (\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta) \ (n!)^2} z^n \quad (\text{II.5}) \]

Taking into account that the Pochhammer k-symbol verified \((\gamma + k)_{n,k} = [1 + n \left( \frac{k}{\gamma} \right)] (\gamma)_{n,k} \) it result

\[ (\gamma + k)_{n,k} - (\gamma)_{n,k} = n \left( \frac{k}{\gamma} \right) (\gamma)_{n,k} \quad (\text{II.6}) \]

Replacing (II.6) in (II.5) it result

\[ W_{k,\alpha,\beta}^{\gamma+k}(z) - W_{k,\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{n \left( \frac{k}{\gamma} \right) (\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta) \ (n!)^2} z^n = \]

\[ \sum_{n=1}^{\infty} \frac{n \left( \frac{k}{\gamma} \right) (\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta) \ (n!)^2} z^n = \sum_{n=0}^{\infty} \frac{(n+1) \left( \frac{k}{\gamma} \right) (\gamma)_{n+1,k}}{\Gamma_k(\alpha n + \alpha + \beta) \ [(n+1)!]^2} z^{n+1} = \]

\[ \sum_{n=0}^{\infty} \frac{\left( \frac{k}{\gamma} \right) (\gamma + k)_{n,k}}{\Gamma_k(\alpha n + \alpha + \beta) \ (n+1)(n!)^2} z \ z^n = k \ z \sum_{n=0}^{\infty} \frac{(\gamma + k)_{n,k}}{\Gamma_k(\alpha n + \alpha + \beta) \ (n+1)(n!)^2} z^n \]

Then, by Lemma 2 it results

\[ W_{k,\alpha,\beta}^{\gamma+k}(z) - W_{k,\alpha,\beta}^{\gamma}(z) = \left( \frac{k}{\gamma} \right) z \frac{d}{dz} W_{k,\alpha,\beta}^{\gamma}(z) \]

In what follows shows the relationship between the k-Wright function and the k-Mittag-Leffler function obtained throug the Laplace transform. In fact, we have the following
Theorem 1 Let $\alpha, \beta, \gamma \in \mathbb{C}$, $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$, $\text{Re}(\gamma) > 0$, $\text{Re}(s) > 0$, $s \neq 0$. Then

$$\mathcal{L} \left[ W_{k,\alpha,\beta}^\gamma(z) \right](s) = \frac{1}{s} E_{k,\alpha,\beta}^\gamma(s^{-1}) \quad \text{(II.7)}$$

where $E_{k,\alpha,\beta}^\gamma(z)$ is the $k$-Mittag-Leffler function given by Dorrego and Cerutti. (cf. [ ]) 

Proof. From definition of Laplace Transform and from (I.3) we have

$$\mathcal{L} \left[ W_{k,\alpha,\beta}^\gamma(z) \right](s) = \int_0^{+\infty} e^{-sz} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} z^n (n!)^2 \, dz =$$

$$\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{1}{(n!)^2} \int_0^{+\infty} e^{-sz} z^n \, dz \quad \text{(II.8)}$$

Taking into account that the integral in (II.8) is

$$\int_0^{+\infty} e^{-sz} z^n \, dz = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \quad \text{(II.9)}$$

From (II.9) and (II.8) we have

$$\mathcal{L} \left[ W_{k,\alpha,\beta}^\gamma(z) \right](s) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{(s^{-1})^{n+1}}{(n!)} =$$

$$\frac{1}{s} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{(s^{-1})^{n}}{(n!)} = \frac{1}{s} E_{k,\alpha,\beta}^\gamma(s^{-1})$$

Then

$$\mathcal{L} \left[ W_{k,\alpha,\beta}^\gamma(z) \right](s) = \frac{1}{s} E_{k,\alpha,\beta}^\gamma(s^{-1})$$

II.2 Fractional Calculus of the $\mathcal{E}(t, k, \alpha, \beta)$ function

In this subsection we will defined the function $\mathcal{E}(t, k, \alpha, \beta)$ in term of the $k$-Wrigth function and then we evaluate its Riemann-Liouville fractional integral and derivative.

Definition 4 Let $\alpha$, $\beta$, $\gamma$ be complex numbers that $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$ and $\text{Re}(\gamma) > 0$, $k > 0$ and $t \in \mathbb{R}$. We define the auxiliary function $\mathcal{E}(t, k, \alpha, \beta)$ by the following relation

$$\mathcal{E}(t, k, \alpha, \beta) = t^{\frac{\alpha}{\beta}} W_{k,\alpha,\beta}^\gamma(t^\alpha) \quad \text{(II.10)}$$
Proposition 1 Let $\alpha, \beta, \gamma, \nu$ be complex numbers that $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$ and $\text{Re}(\nu) > 0$, $k > 0$ and $t \in \mathbb{R}$ then

$$I_+^\nu (\mathcal{E}(t, k, \alpha, \beta)) (x) = k^\nu x^{-\frac{\alpha}{k} + \nu - 1} W_\nu^\gamma (x^\frac{\alpha}{k}) \tag{II.11}$$

Proof. By definition (I.1) and (I.3) we have

$$I_+^\nu (\mathcal{E}(t, k, \alpha, \beta)) (x) = \frac{1}{\Gamma(\nu)} \int_0^x t^{\frac{\alpha}{k} - 1} \sum_{n=0}^{\infty} \frac{\Gamma(n, \beta)}{\Gamma(n + \beta) (n!)^2} (x - t)^{\nu - 1} dt =$$

$$= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(n, \beta)}{\Gamma(n + \beta) (n!)^2} \int_0^x t^{\frac{\alpha + \beta}{k} - 1} (x - t)^{\nu - 1} dt \tag{II.12}$$

making the change of variable

$$t = \xi x, \quad x - t = x(1 - \xi), \quad dt = xd\xi$$

and replacing in (II.12) it result

$$I_+^\nu (\mathcal{E}(t, k, \alpha, \beta)) (x) = \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(n, \beta)}{\Gamma(n + \beta) (n!)^2} \int_0^1 (\xi x)^{\frac{\alpha + \beta}{k} - 1} [x(1 - \xi)]^{\nu - 1} xd\xi =$$

$$= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(n, \beta)}{\Gamma(n + \beta) (n!)^2} x^{\frac{\alpha + \beta}{k} + \nu - 1} \int_0^1 \xi^{\frac{\alpha + \beta}{k} - 1} (1 - \xi)^{\nu - 1} d\xi \tag{II.13}$$

The integral in (II.13) result

$$\int_0^1 \xi^{\frac{\alpha + \beta}{k} - 1} (1 - \xi)^{\nu - 1} d\xi = B \left( \frac{\alpha + \beta}{k}, \nu \right)$$

where $B(z, w)$ is the Beta function. Then

$$I_+^\nu (\mathcal{E}(t, k, \alpha, \beta)) (x) = \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(n, \beta)}{\Gamma(n + \beta) (n!)^2} x^{\frac{\alpha + \beta}{k} + \nu - 1} B \left( \frac{\alpha + \beta}{k}, \nu \right) =$$

$$= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(n, \beta)}{\Gamma(n + \beta) (n!)^2} x^{\frac{\alpha + \beta}{k} + \nu - 1} \frac{\Gamma \left( \frac{\alpha + \beta}{k} \right) \Gamma(\nu)}{\Gamma \left( \frac{\alpha + \beta}{k} + \nu \right)} =$$

$$= \frac{1}{\Gamma(\nu)} \sum_{n=0}^{\infty} \frac{\Gamma(n, \beta)}{\Gamma(n + \beta) (n!)^2} x^{\frac{\alpha + \beta}{k} + \nu - 1} \frac{\Gamma \left( \frac{\alpha + \beta}{k} \right) \Gamma(\nu)}{\Gamma \left( \frac{\alpha + \beta + \nu}{k} \right)}$$
\[
\frac{1}{\Gamma(\nu)} x^{\frac{\alpha}{k} + \nu - 1} k^\nu \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k} (x_k^\alpha)^n}{\Gamma_k (\alpha n + \beta) \Gamma(\nu)} \Gamma_k (\alpha n + \beta + k \nu) = k^\nu x^{\frac{\alpha}{k} + \nu - 1} W^\gamma_{k,\alpha,\beta + k \nu} (x_k^\alpha)
\]

Proposition 2 Let \(\alpha, \beta, \gamma, \nu\) be complex number that Re\((\alpha) > 0\), Re\((\beta) > 0\) and \(r - 1 < \text{Re}(\nu) \leq r\), \(r \in \mathbb{N}\), \(k > 0\) and \(t \in \mathbb{R}\) then

\[
D^\nu_+ (E(t, k, \alpha, \beta))(x) = k^{-\nu} x^{\frac{\alpha r + \beta}{k} + r - \nu - 1} (\gamma)_{r,k} \sum_{n=0}^{\infty} \frac{(\gamma + k)_{n,k}}{\Gamma_k (\alpha(n + r) + \beta - k \nu)} \left(\frac{x_k^\alpha}{(n + r)!}\right)^n (\nu = 1)
\]

Proof. The Riemann-Liouville fractional integral of order \(r - \nu\) of \(E(t, k, \alpha, \beta)\) result

\[
I^{r-\nu}_+ (E(t, k, \alpha, \beta))(x) = k^{r-\nu} x^{\frac{\alpha r + \beta}{k} - 1} W^\gamma_{k,\alpha,\beta + (r-\nu)k} (x_k^\alpha) \quad \text{(II.15)}
\]

Now we evaluate the derivative of order \(r\) with respect to \(x\) of (II.15)

\[
\frac{d^r}{dx^r} k^{r-\nu} x^{\frac{\alpha r + \beta}{k} + r - 1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k (\alpha(n + r) + \beta - k \nu)} (n!) =
\]

\[
\frac{d^r}{dx^r} k^{r-\nu} \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k (\alpha n + \beta + (r - \nu)k)} x^{\frac{\alpha n + \beta + (r - \nu)k}{k} - 1} =
\]

\[
k^{r-\nu} \sum_{n \geq r} (\gamma)_{n,k} \frac{\left(\frac{\alpha n + \beta}{k} + r - \nu - 1\right) \ldots \left(\frac{\alpha n + \beta}{k} - (r - \nu) - 1\right)}{\Gamma_k (\alpha n + \beta + (r - \nu)k)} (n!) =
\]

\[
k^{r-\nu} \sum_{n \geq r} \frac{(\gamma)_{n,k}}{k^{\frac{\alpha n + \beta}{k} + (r - \nu)k}} \Gamma\left(\frac{\alpha n + \beta + (r - \nu)k}{k} - 1\right) (n!) \Gamma\left(\frac{\alpha n + \beta - k \nu}{k} - 1\right) =
\]

\[
k^{r-\nu} \sum_{n \geq r} \frac{(\gamma)_{n,k}}{k^{\frac{\alpha n + \beta - k \nu}{k}} \Gamma\left(\frac{\alpha n + \beta - k \nu}{k} - 1\right) (n!)^2} =
\]

\[
k^{r-\nu} x^{\frac{\alpha}{k} + r - 1} \sum_{n \geq r} \frac{(\gamma)_{n,k}}{\Gamma_k (\alpha n + \beta - k \nu) (n!)} =
\]

\[
k^{r-\nu} x^{\frac{\alpha}{k} + r - 1} \sum_{n \geq r} \frac{(\gamma)_{n,k}}{\Gamma_k (\alpha n + \beta - k \nu) (n!^2)} =
\]
\[ k^{-\nu} x^{\frac{\alpha}{k} - \nu - 1} \sum_{n=0}^{\infty} \frac{(\gamma)_{n+r,k} x^{n+\alpha} x^{\beta - k\nu}}{\Gamma_k(\alpha + \alpha n + \beta - k\nu) [(n + r)!]^2} \] (II.16)

and using the relation \((\gamma)_{n+r,k} = (\gamma)_{r,k}(\gamma + k)_{n,k}\), (II.16) result

\[ D_+^\nu (\mathcal{E}(t, k, \alpha, \beta))(x) = k^{-\nu} x^{\frac{\alpha \nu + \beta}{k} - \nu - 1} \sum_{n=0}^{\infty} \frac{(\gamma + k)_{n,k} x^n}{\Gamma_k(\alpha(n + r) + \beta - k\nu) [(n + r)!]^2} \]

\[ \blacksquare \]

**Proposition 3** Let \(\alpha, \beta, \gamma, \nu\) be complex number that \(\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(\gamma) > 0\) and \(j \in \mathbb{N}\). Then

\[ \left( \frac{d}{dt} \right)^j \mathcal{E}(t, k, \alpha, \beta) = k^{-j} x^{\frac{(\alpha - k) + \beta}{k} - 1} (\gamma)_{j,k} \sum_{n=0}^{\infty} \frac{(\gamma + k)_{n,k} x^n}{\Gamma_k(\alpha(n + \beta + j(\alpha - k))) [(n + j)!]^2} \]

\[ \text{(II.17)} \]

**Proof.** It is sufficient take \(\nu = r = j\) in Proposition 2. \(\blacksquare\)

**References**


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