On Higher Order Fractional Bessel Potentials

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Abstract

The solutions of the equation

\[(I-\Delta)^{\alpha/2} u_{k,\alpha}(x) = f(x), \quad k \geq 1, \alpha > 0 \quad \text{in} \quad \mathbb{R}^n\]

where \( f \in L^2(\mathbb{R}^n) \) are investigated, fractional Bessel kernel of higher order are defined, and recurrence relations between these solutions and fractional Bessel kernel are obtained. Finally explicit formulas for the solutions when \( x \in \mathbb{R}^n, \quad m=0,1 \) are given.

Keywords: Fractional Bessel potential of higher order, fractional Bessel kernel, higher order, recurrence relations, explicit formulas

1. Introduction

This work is an extension to [1], [2] where the case \( \alpha = 2 \) was considered. We consider first \( k = 1 \), i.e.,

\[(I-\Delta)^{\alpha/2} u_{1,\alpha}(x) = f(x), \quad \alpha > 0 \quad \text{in} \quad \mathbb{R}^n\]

Then,

\[u_{1,\alpha}(x) = (I-\Delta)^{-\alpha/2} f(x) = B_{1,\alpha}(x) * f(x)\]

where from [3], [4], [5] the \( B_{1,\alpha}(x) \) fractional Bessel kernel of first order with \( \alpha > 0 \) is defined as follows

**Definition**

\[B_{1,\alpha}(x) = \left( \frac{1}{(4\pi)^{\alpha/2} \Gamma(\alpha/2)} \right) \int_0^\infty \exp(-\pi |x|^2/t) \exp(-t/4\pi) t^{-(\alpha+1)/2} \frac{dt}{t}\]
Also, we have the following Proposition

**Proposition**

For the $B_{1,\alpha}(x)$ with $\alpha > 0$, the following relations are true

**I-**

$$B_{1,\alpha}(x) \in L^1(\mathbb{R}^n)$$

**II-**

$$\hat{B}_{1,\alpha}(x) = \left(1 + 4\pi^2 |x|^2\right)^{-\alpha/2}$$

which can be verified by application of Fourier transform [4], and by using Fubini's theorem, where the Fourier transform of a function $f$ is given by

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x t} f(t) \, dt$$

Then from Eq. (4), (5), (6), and convolution theorem, we have

$$u_{1,\alpha}(x) = \left(\frac{1}{\gamma(\alpha)}\right) \int_{\mathbb{R}^n} \left[ \int_0^{\infty} \exp\left(-\pi|x-y|^2/t\right) \exp\left(-t/4\pi\right) t^{-(n+\alpha)/2} \, f(y) \, dt \right] f(y) \, dy$$

$$= \int_{\mathbb{R}^n} B_{1,\alpha}(x-y) f(y) \, dy$$

(7)

Where $\alpha > 0$, and $\gamma(\alpha) = (4\pi)^{n/2} \Gamma(\alpha/2)$. Then we have the following result

**Theorem (1)**

Solution $u_{1,\alpha}(x)$ of Eq. (2) for $\alpha > 0$ in $\mathbb{R}^n$ is Eq. (7)

2. **The solutions** $u_{2,\alpha}(x)$

Where $k = 2$ in Eq. (1), we have

$$(I-\Delta)^2 u_{2,\alpha}(x) = f(x) \quad \text{in} \quad \mathbb{R}^n \quad \text{where} \quad \alpha > 0, \quad f \in L^2(\mathbb{R}^n),$$

and we can write this equation in the form

$$(I-\Delta)^2 u_{2,\alpha}(x) = u_{1,\alpha}(x)$$

(9)

Then from Eq. (9), and Eq. (7), we get

$$u_{2,\alpha}(x) = \left(\frac{1}{\gamma(\alpha)}\right) \int_{\mathbb{R}^n} \left[ \int_0^{\infty} \exp\left(-\pi|x-y|^2/t\right) \exp\left(-t/4\pi\right) t^{-(n+\alpha)/2} u_{1,\alpha}(y) \, dy \, dt \right] f(y) \, dy$$

(10)

by rearranging the order of integration, we get

$$u_{2,\alpha}(x) = \left(\frac{1}{\gamma(\alpha)}\right) \int_{\mathbb{R}^n} \left[ \int_0^{\infty} \exp\left(-\pi|x-y|^2/\tau\right) \exp\left(-\tau/4\pi\right) \tau^{-(n+\alpha)/2} \, dt \times \int_0^{\infty} \exp\left(-\pi|\tilde{y}|^2/\tau\right) \exp\left(-\tau/4\pi\right) \tau^{-(n+\alpha)/2} \, d\tilde{y} \right] f(y) \, dy$$

$$= \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} B_{1,\alpha}(x-\tilde{y}) B_{1,\alpha}(\tilde{y} - y) \, d\tilde{y} \right] f(y) \, dy$$

Thus, $u_{2,\alpha}(x) = \int_{\mathbb{R}^n} B_{2,\alpha}(x,y) f(y) \, dy = B_{2,\alpha}(x,y) \ast f(x)$

(11)
where
\[ B_{2,\alpha}(x, y) = \int_{\mathbb{R}^n} B_{1,\alpha}(x - \tilde{y}) \, B_{1,\alpha}(y - \tilde{y}) \, d\tilde{y} \]  
(12)

\( B_{2,\alpha} \) is called the Fractional Bessel kernel of second order. Then from Eq. (11) and Eq. (12), we have the following result

**Theorem (2)**

Solution \( u_{2,\alpha}(x) \) of Eq. (8) for \( \alpha > 0 \) in \( \mathbb{R}^n \) is Eq. (11), (12)

### 3. Recurrence relations and fractional Bessel kernel of order \( k \)

**Theorem (3)**

For all \( k \geq 2 \) the solution of equation

\[ (I - \Delta)^{\alpha/2} u_{k,\alpha}(x) = f(x) \quad \alpha > 0 \quad \text{in} \quad \mathbb{R}^n \]

(13)
is

\[ u_{k,\alpha}(x) = \int_{\mathbb{R}^n} B_{k,\alpha}(x, y) \, f(y) \, dy = B_{k,\alpha}(x, y) \ast f(x) \]

(14)

where

\[ B_{k,\alpha}(x, y) = \int_{\mathbb{R}^n} B_{1,\alpha}(x - \tilde{y}) \, B_{k-1,\alpha}(\tilde{y}, y) \, d\tilde{y} \quad , \quad k \geq 3 \]

(15)
is called fractional Bessel kernel of order \( k \).

**Proof**

The relations (13), (14) and (15) are true for \( k = 2 \). Let these relations are true for some \( k = m \), i.e.,

\[ u_{m,\alpha}(x) = B_{m,\alpha}(x, y) \ast f(x) \]

(16)
is the solution of equation

\[ (I - \Delta)^{\alpha/2} u_{m,\alpha}(x) = f(x) \quad \alpha > 0 \quad \text{in} \quad \mathbb{R}^n \]

(17)

We now try to prove that relations (13) and (14) are true for some \( k = m + 1 \).

For \( k = m + 1 \), we can write Eq. (13) in the following form

\[ (I - \Delta)^{\alpha/2} u_{m+1,\alpha}(x) = f(x) \quad \alpha > 0 \quad \text{in} \quad \mathbb{R}^n \]

(18)

which equivalent to the following system

\[ (I - \Delta)^{\alpha/2} u_{m+1,\alpha}(x) = W(x) \]

(19)

\[ (I - \Delta)^{\alpha/2} W(x) = f(x) \]

(20)

From Eq. (20), and since the relations (13) and (14) are true for some \( k = m \), we get

\[ W(x) = u_{m,\alpha}(x) = B_{m,\alpha}(x, y) \ast f(x) = \int_{\mathbb{R}^n} B_{m,\alpha}(x, y) \, f(y) \, dy \]

(21)

where

\[ B_{m,\alpha}(x, y) = \int_{\mathbb{R}^n} B_{1,\alpha}(x - \tilde{y}) \, B_{m-1,\alpha}(\tilde{y}, y) \, d\tilde{y} \quad , \quad m \geq 3 \]

(22)

By substituting from Eq. (21) in Eq. (19) and using (7), we get
Using (21), (22), (4) and then rearrange the order of integration, we get

\[ u_{m+1,\alpha}(x) = \frac{1}{\gamma(\alpha)} \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}^n} \exp\left( -\pi |x - \tilde{y}|^2 \right) \exp(-t/4\pi) t^{(-\alpha+1)/2} u_{\alpha,\alpha}(\tilde{y}) d\tilde{y} dt \right) \]

\[ = \int_{\mathbb{R}^n} M_{m+1,\alpha}(x, y) f(y) dy \]

where

\[ B_{m+1,\alpha}(x, y) = \int_{\mathbb{R}^n} B_{1,\alpha}(x - \tilde{y}) B_{m,\alpha}(\tilde{y}, y) d\tilde{y} \]

(23)

Then relation (14) and (15) are true for \( k = m + 1 \).

**Theorem (4)**

The fractional Bessel kernel \( B_{k,\alpha}(x, y), \ k \geq 2 \) is symmetric in the variables, i.e. \( B_{k,\alpha}(x, y) = B_{k,\alpha}(y, x) \).

**Proof**

The proof follows immediately using (12), (15), (24) using mathematical induction.

**Theorem (5)**

The fractional Bessel kernel \( B_{k,\alpha}(x, y) \) satisfies

\[ (u_{r,\alpha} * B_{1,\alpha}) * B_{k-r-1,\alpha} = u_{r,\alpha} * (B_{1,\alpha} * B_{k-r-1,\alpha}) \]

for \( 1 \leq r \leq k - 1 \)

**Proof**

Using (14), (24) and Th. (4), then

\[ (u_{r,\alpha} * B_{1,\alpha}) * B_{k-r-1,\alpha} = \int_{\mathbb{R}^n} B_{1,\alpha}(x - \tilde{y}) u_{r,\alpha}(\tilde{y}) d\tilde{y} * B_{k-r-1,\alpha}(x, y) \]

\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} B_{k-r-1,\alpha}(x, y) B_{1,\alpha}(y - \tilde{y}) u_{r,\alpha}(\tilde{y}) d\tilde{y} dy \]

\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} B_{k-r-1,\alpha}(x, \tilde{y}) B_{1,\alpha}(\tilde{y} - y) d\tilde{y} u_{r,\alpha}(y) dy \]

\[ = u_{r,\alpha}(x) * B_{k-r-1,\alpha}(x, y) = u_{r,\alpha} * (B_{k-r-1,\alpha} * B_{1,\alpha}) \]

**Theorem (6)**

For all \( k \geq 1 \) the equation

\[ (I - \Delta)^{\alpha/2} u_{k,\alpha}(x) = f(x), \ \alpha > 0 \quad \text{in} \quad \mathbb{R}^n \]

(25)

is uniquely solvable by

\[ \text{I-} \quad u_{k,\alpha}(x) = u_{k-1,\alpha} * B_{1,\alpha} \quad \text{For} \ k \geq 2 \]

(26)

\[ \text{or} \quad u_{k,\alpha}(x) = u_{k-2,\alpha} * B_{2,\alpha} \quad \text{For} \ k \geq 3 \]

(27)

\[ \text{II-Moreover, if} \ k \geq 2, \ \text{we have} \]

\[ B_{k,\alpha} * f = u_{r,\alpha} * B_{k-r,\alpha} \quad \text{For} \ 1 \leq r \leq k - 1 \]

(28)

**Proof**

I-We can write Eq. (25) in the form...
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\[(I-\Delta)^{x/2}u_{k,\alpha}(x) = W(x)\]  \hspace{1cm} (29)
\[(I-\Delta)^{x(k-1)/2}W(x) = f(x)\]  \hspace{1cm} (30)

From Eq. (30), we get
\[W(x) = u_{k-1,\alpha}(x)\]  \hspace{1cm} (31)

By substituting from Eq. (31) in Eq. (29) and using Eq. (3), and rearrange the order of integration, we get directly for \(k \geq 2\), that \(u_{k,\alpha}(x) = u_{k-1,\alpha} * B_{1,\alpha}\), Thus the first part of (I) is proven. We can write Eq. (25) in the form
\[(I-\Delta)^{2\alpha/2}u_{k,\alpha}(x) = W(x)\]  \hspace{1cm} (32)
\[(I-\Delta)^{\alpha(k-2)/2}W(x) = f(x)\]  \hspace{1cm} (33)

From Eq. (33), we get
\[W(x) = u_{k-2,\alpha}(x)\]  \hspace{1cm} (34)

By substituting from Eq. (34) in Eq. (32) and using Eq. (11),(12), we get directly for \(k \geq 3\), that \(u_{k,\alpha}(x) = u_{k-2,\alpha} * B_{2,\alpha}\), Thus the second part of (I) is proven.

II-By writing Eq. (25) in the form
\[(I-\Delta)^{x/r^2}(I-\Delta)^{(k-r)/2}u_{k,\alpha}(x) = f(x)\]
Which equivalent to the following system
\[(I-\Delta)^{(k-r)/2}u_{k,\alpha}(x) = W(x)\]  \hspace{1cm} (35)
\[(I-\Delta)^{x/r^2}W(x) = f(x)\]  \hspace{1cm} (36)

From Eq. (36), we get
\[W(x) = u_{r,\alpha}(x)\]  \hspace{1cm} (37)

By substituting from Eq. (37) in Eq. (35), we have
\[(I-\Delta)^{(k-r)/2}u_{k,\alpha}(x) = u_{r,\alpha}(x)\, ,\]
and then by using Eq. (26),for \(k \geq 2\, ,\) we can prove as follows ,for \(1 \leq r \leq k-1\, ,\) that
\[u_{r,\alpha}(x) = u_{r,\alpha} * B_{k-r,\alpha}\]  \hspace{1cm} (38)

Let it is true for some \(r\, ,\) and we prove for \(r+1\, ,\) for \(1 \leq r+1 \leq k-1\, ,\) i.e. we prove that \(u_{k,\alpha}(x) = u_{r+1,\alpha} * B_{k-r-1,\alpha}\). Using theorems (6)(I),(5) and (3),(7),(12),(14),(15) we obtain
\[u_{k,\alpha}(x) = (u_{r,\alpha} * B_{1,\alpha} ) * B_{k-r-1,\alpha} = u_{r,\alpha} * B_{k-r,\alpha} = B_{k,\alpha} * f\]
and (II) is proven.

4. Explicit formula for the solution \(u_{k,\alpha,m}(x)\) of:
\[(I-\Delta)^{x/k}u_{k,\alpha,m}(x) = [k]^m \text{ in } \Re^n , \, k \geq 1, \, \alpha > 0, \, m = 0, 1\]  \hspace{1cm} (39)
\(1^k - k = 1, m = 0\) Then
\[(I-\Delta)^{x/2}u_{1,\alpha,0}(x) = 1\]  \hspace{1cm} (40)

And
Let \( r = y - x \Rightarrow dr = dy \), then

\[
\int_{\mathbb{R}^n} \exp\left(-\pi |x-y|^2/t\right) dy = \int_{\mathbb{R}^n} \exp\left(-\pi |r|^2/t\right) dr
\]

\[
= \prod_{i=1}^{n} \int_{-\infty}^{\infty} \exp\left(-\pi r_i^2/t\right) dr_i = t^{n/2}
\]

By substituting from Eq. (42) in Eq. (41), we get

\[
u_{1,\alpha,0}(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \exp\left(-\pi |r|^2/t\right) t^{\alpha/2} dt
\]

But from definition of gamma function, we have

\[
\left(\frac{1}{4\pi}\right)^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)} \int_{0}^{\infty} \exp(-t/4\pi) t^{\alpha/2} dt
\]

Then

\[
u_{1,\alpha,0}(x) = 1
\]

\[
2^\alpha - k = 2, m = 0 \text{ Then}
\]

\[
(I-\Delta)^k \nu_{2,\alpha,0}(x) = 1
\]

\[
(I-\Delta)^{\alpha/2} (I-\Delta)^{\alpha/2} \nu_{2,\alpha,0}(x) = 1
\]

By write above equation in the equivalent system

\[
(I-\Delta)^{\alpha/2} w = 1
\]

\[
(I-\Delta)^{\alpha/2} \nu_{2,\alpha,0}(x) = w
\]

From Eq. (46) and Eq. (40), we have directly

\[
w = \nu_{1,\alpha,0}(x) = 1
\]

By substituting from Eq. (48) in Eq. (47), we get

\[
(I-\Delta)^{\alpha/2} \nu_{2,\alpha,0}(x) = 1
\]

\[
u_{2,\alpha,0}(x) = (I-\Delta)^{-\alpha/2}(1) = B_{1,\alpha,0}(x) * 1 = \int_{\mathbb{R}^n} B_{1,\alpha,0}(x-y) dy
\]

Then

\[
u_{2,\alpha,0}(x) = 1
\]

\[
3^\alpha - m = 0, k \geq 1 \text{ Then}
\]

**Theorem (7)**

For all \( k \geq 1, \alpha > 0, m = 0 \) the solution \( \nu_{k,\alpha,0}(x) \) of (39) is

\[
u_{k,\alpha,0}(x) = 1
\]
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Proof

Let the relations (39) and (50) be true for some \( k = r \), i.e.

\[
u_{r,\alpha,0}(x) = 1
\]  

is the solution of equation

\[
(I - \Delta)^{\alpha r/2} u_{r,\alpha,0}(x) = 1, \quad \alpha > 0 \quad \text{in} \quad \mathbb{R}^n
\]  

We now try to prove for \( k = r + 1 \), so we can write Eq. (39) in the following form

\[
(I - \Delta)^{\alpha (r+1)/2} u_{r+1,\alpha,0}(x) = 1, \quad \alpha > 0 \quad \text{in} \quad \mathbb{R}^n
\]  

which is equivalent to the following system

\[
(I - \Delta)^{\alpha r/2} u_{r+1,\alpha,0}(x) = W(x)
\]

\[
(I - \Delta)^{\alpha r/2} W(x) = 1
\]

Then using (51), we get

\[
W(x) = u_{r,\alpha,0}(x) = 1
\]  

by substituting from Eq. (56) in Eq. (54), we get

\[
(I - \Delta)^{\alpha r/2} u_{r+1,\alpha,0}(x) = 1
\]

\[
u_{r+1,\alpha,0}(x) = (I - \Delta)^{\alpha r/2}(1) = B_{1,\alpha,0}(x)*1 = \int_{\mathbb{R}^n} B_{1,\alpha,0}(x - y) dy
\]

Then

\[
u_{r+1,\alpha,0}(x) = 1
\]  

4° \( k = 1, m = 1 \) Then

\[
(I - \Delta)^{\alpha r/2} u_{1,\alpha,1}(x) = \vert x \vert^2
\]  

Then

\[
u_{1,\alpha,1}(x) = (I - \Delta)^{-\alpha r/2}(\vert x \vert^2)
\]

\[
= \frac{1}{\gamma(\alpha) \mathbb{R}^n} \int_{0}^{\infty} \int_{\mathbb{R}^n} \exp(-\pi \vert x - y \vert^2 / t) \exp(-t / 4 \pi) t^{-\alpha r - \alpha / 2} \vert y \vert^2 \frac{dt}{t} dy
\]  

Let

\[
y - x = (\sqrt{t} / \sqrt{\pi}) z \Rightarrow \frac{-\pi}{t} \vert y - x \vert^2 = -\vert z \vert^2
\]

\[
\vert y \vert^2 = \vert x \vert^2 + (2 \sqrt{t} / \sqrt{\pi}) x.z + (t / \pi) \vert z \vert^2
\]

Then

\[
u_{1,\alpha,1}(x) = A + B + C
\]  

Where

\[
A = \frac{\vert x \vert^2}{\gamma(\alpha) \mathbb{R}^n} \int_{0}^{\infty} \int_{\mathbb{R}^n} \exp(-\pi \vert z \vert^2) \exp(-t / 4 \pi) t^{-\alpha r - \alpha / 2} \frac{dt}{t} \left( \frac{\sqrt{t}}{\sqrt{\pi}} \right)^n dz
\]

By rearranging the order of integration, we get
\[ A = \frac{|x|^2}{\gamma(\alpha)} \int_0^\infty \exp\left(-t/4\pi\right) t^{\alpha/2} \frac{dt}{t} \int_{\mathbb{R}^n} \frac{1}{(\sqrt{\pi})^n} \exp\left(-|z|^2\right) dz \]
\[ = \frac{|x|^2}{\gamma(\alpha)} \int_0^\infty \exp\left(-t/4\pi\right) t^{\alpha/2} \frac{dt}{t} = |x|^2 \quad (61) \]

And
\[ B = \frac{1}{\gamma(\alpha)} \int_0^\infty \exp\left(-|z|^2\right) \exp\left(-t/4\pi\right) t^{-(\alpha+1)/2} \frac{2\sqrt{t}}{\sqrt{\pi}} (x,z) \frac{dt}{t} \left(\frac{\sqrt{t}}{\sqrt{\pi}}\right)^n dz \]
\[ = 2\pi^{-1/2} \frac{1}{\gamma(\alpha)} \int_0^\infty \exp\left(-|z|^2\right) \exp\left(-t/4\pi\right) t^{-(\alpha+1)/2} (x,z) \frac{dt}{t} \left(\frac{\sqrt{t}}{\sqrt{\pi}}\right)^n dz \]
\[ = 2\left(\frac{\pi}{\gamma(\alpha)}\right)^{-1/2} \int_0^\infty \exp\left(-t/4\pi\right) t^{(\alpha+1)/2} \frac{dt}{t} \int_{\mathbb{R}^n} \exp\left(-|z|^2\right) (x,z) dz \]

But since
\[ \int_{\mathbb{R}^n} \exp\left(-|z|^2\right) (x,z) dz = \sum_{i=1}^n x_i z_i e^{-i\frac{\pi}{4}} = 0. \]

Then
\[ B = 0 \quad (62) \]

Finally
\[ C = \frac{1}{\gamma(\alpha)} \int_0^\infty \exp\left(-|z|^2\right) \exp\left(-t/4\pi\right) t^{-(\alpha+1)/2} \frac{1}{\pi} |z|^2 \frac{dt}{t} \left(\frac{\sqrt{t}}{\sqrt{\pi}}\right)^n dz \]
\[ = \frac{\pi}{\gamma(\alpha)} \int_0^\infty \exp\left(-t/4\pi\right) t^{(\alpha+2)/2} \frac{dt}{t} \int_{\mathbb{R}^n} \exp\left(-|z|^2\right) dz , \]

but since we can deduce
\[ \int_{\mathbb{R}^n} \exp\left(-|z|^2\right) dz = \frac{1}{2} n \pi^{n/2}, \]

also using (43) we obtain that
\[ C = \alpha n \quad (63) \]

then we get
\[ u_{1,\alpha,1}(x) = |x|^2 + \alpha n \quad (64) \]

5° - \( m = 1, \ k = 2 \) Then
\[ (I-\Delta)^{\alpha/2} u_{2,\alpha,1}(x) = |x|^2 \quad (65) \]
\[ (I-\Delta)^{\alpha/2} u_{2,\alpha,1}(x) = |x|^2 \]

By write above equation in the equivalent system
\[ (I-\Delta)^{\alpha/2} w = |x|^2 \quad (66) \]
\[ (I-\Delta)^{\alpha/2} u_{2,\alpha,1}(x) = w \quad (67) \]

From Eq. (66) and Eq. (58), we have directly
\[ w = u_{1,\alpha,1}(x) = |x|^2 + \alpha n \quad (68) \]
By substituting from Eq. (68) in Eq. (67), we get
\[
(I - \Delta)^{\alpha/2} u_{2,\alpha,1}(x) = |x|^2 + \alpha n
\]
\[
u_{2,\alpha,1}(x) = (I - \Delta)^{-\alpha/2}(|x|^2 + \alpha n) = B_{1,\alpha,0}(x) \frac{|x|^2 + \alpha n}{\gamma(\alpha) \Gamma(\alpha)}
\]
\[
= \frac{1}{\gamma(\alpha) \Gamma(\alpha)} \int_0^\infty \exp\left(-\pi |x-y|^2 / t\right) \exp(-t/4\pi) t^{-\alpha/2} |y|^2 dt dy
\]
\[
+ \frac{\alpha n}{\gamma(\alpha) \Gamma(\alpha)} \int_0^\infty \exp\left(-\pi |x-y|^2 / t\right) \exp(-t/4\pi) t^{(-\alpha+1)/2} dt dy
\]
\[
= \frac{1}{\gamma(\alpha) \Gamma(\alpha)} \int_0^\infty \exp\left(-\pi |x-y|^2 / t\right) \exp(-t/4\pi) t^{(-\alpha+1)/2} |y|^2 dt dy
\]
\[
+ \alpha n \int_0^\infty B_{1,\alpha,0}(x-y) dy
\]
And by using Eq. (41) and Eq. (44), we get
\[
u_{2,\alpha,1}(x) = \frac{1}{\gamma(\alpha) \Gamma(\alpha)} \int_0^\infty \exp\left(-\pi |x-y|^2 / t\right) \exp(-t/4\pi) t^{(-\alpha+1)/2} |y|^2 dt dy + \alpha n
\]
And by using Eq. (59) and Eq. (64), we deduce
\[
u_{2,\alpha,1}(x) = |x|^2 + 2\alpha n = u_{1,\alpha,1}(x) + \alpha n
\]
(69)
6' - m = 1, k \geq 1 Then
We expect and prove the next theorem

**Theorem (8)**

For all $k \geq 1, \alpha > 0, m = 1$ the solution $u_{k,\alpha,1}(x)$ of (39) is
\[
u_{k,\alpha,1}(x) = u_{k-1,\alpha,1}(x) + \alpha n
\]
(70)
**Proof**

Let the relations (39) and (70) are true for some $k = r$ i. e.,
\[
u_{r,\alpha,1}(x) = u_{r-1,\alpha,1}(x) + \alpha n
\]
(71)
is the solution of equation
\[
(I - \Delta)^{\alpha/2} u_{r,\alpha,1}(x) = |x|^2, \quad \alpha > 0 \quad in \quad \mathbb{R}^n
\]
(72)
For $k = r + 1$, we can write Eq. (39) in the following form
\[
(I - \Delta)^{\alpha(r+1)/2} u_{r+1,\alpha,1}(x) = |x|^2, \quad \alpha > 0 \quad in \quad \mathbb{R}^n
\]
(73)
which equivalent to the following system
\[
(I - \Delta)^{\alpha/2} u_{r+1,\alpha,1}(x) = W(x)
\]
(74)
\[
(I - \Delta)^{\alpha(r+1)/2} W(x) = |x|^2
\]
(75)
and then using (71) we get
\[
W(x) = u_{r-1,\alpha,1}(x) + \alpha n
\]
(76)
Then by substituting from Eq. (76) in Eq. (74), we get
\[
(I - \Delta)^{\alpha/2} u_{r+1,\alpha,1}(x) = u_{r-1,\alpha,1}(x) + \alpha n
\]
then using (7), we get
\[ u_{r+1,\alpha,1}(x) = \frac{1}{\gamma(\alpha) R^n} \int_0^\infty \int_0^\infty \exp\left(-\pi |x-y|^2/t\right) \exp\left(-t/4\pi\right) t^{(-n+\alpha)/2} u_{r-1,\alpha,1}(y) \frac{dt}{t} \frac{dy}{y} + \frac{\alpha n}{\gamma(\alpha) R^n} \int_0^\infty \int_0^\infty \exp\left(-\pi |x-y|^2/t\right) \exp\left(-t/4\pi\right) t^{(-n+\alpha)/2} \frac{dt}{t} \frac{dy}{y} \]
\[ = u_{r,\alpha,1}(x) + \alpha n \int_{\mathbb{R}^n} B_{1,\alpha,0}(x-y) \, dy = u_{r,\alpha,1}(x) + \alpha n \quad (77) \]

Then the proof is complete.

References


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