A Class of Univalent Harmonic Meromorphic Functions with Respect to k-Symmetric Points

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Abstract

After reading so many articles with respect to symmetric points such as [1], [6], [7], [8] in analytic functions, meromorphic functions, harmonic functions and harmonic meromorphic functions in punctured disk. In this article, I have an idea to introduce a class of univalent harmonic meromorphic functions with respect to k-symmetric points in outside of an unit disk. Some properties like coefficient condition, bounds and extreme points for functions belongs to the class has been studied.

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1 Introduction

A continuous function \( f = u + iv \), is a complex-valued harmonic function in a domain \( D \subset C \), if both \( u \) and \( v \) are real harmonic in \( D \). Cluine and Sheil-small [2] investigated the family of all complex-valued harmonic mappings \( f \) defined on the open unit disk \( U \), which admits the representation \( f(z) = h(z) + g(z) \) where \( h \) and \( g \) are analytic univalent in \( U \).

Hengartner and Schober [3] considered the class \( \sum_H \) of functions which are harmonic, meromorphic, orientation-preserving and univalent in \( \tilde{U} = \{ z : |z| > 1 \} \) so that \( f(\infty) = \infty \). Such functions admit the representation

\[
 f(z) = h(z) + g(z), \tag{1}
\]
where
\[ h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n} \] (2)
are analytic in \( \tilde{U} = \{ z = |z| > 1 \} \).

Analogous to the concept given by Sakaguchi [5] for the class \( S^*_s \) of functions \( f(z) \in S \), which are starlike with respect to symmetric points, the definition may be extended for harmonic meromorphic functions defined as follows:

**Definition 1.1** A function \( f(z) \in \sum_H \) is said to be in the class \( \sum_{H^*} \) of starlike functions with respect to symmetric points, if it satisfies
\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0,
\]
where \( f_2(z) = \frac{1}{2} [f(z) - f(-z)] \).

**Definition 1.2** A function \( f(z) \in \sum_H \) is said to be in the class \( \sum_{H^k} \) of starlike functions with respect to symmetric points, if it satisfies
\[
\text{Re} \left\{ \frac{zf'(z)}{f_k(z)} \right\} > 0,
\]
where
\[
f_k(z) = \frac{1}{k} \sum_{j=0}^{k-1} \epsilon^{-j} f(\epsilon^j z), \quad \epsilon = \exp\left(\frac{2\pi i}{k}\right), \quad k \geq 2.
\]

Denote \( \sum_{H^1} \equiv \sum_H, \quad \sum_{H^2} \equiv \sum_{H^*} \).

**Definition 1.3** A function \( f(z) \in \sum_H \) is said to be in the class \( \sum_{H^k}(\alpha), \ (0 \leq \alpha < 1) \) of starlike functions with respect to symmetric points, if it satisfies
\[
\text{Re} \left\{ \frac{zf'(z)}{f_k(z)} \right\} > \alpha,
\]
where
\[
f_k(z) = \frac{1}{k} \sum_{j=0}^{k-1} \epsilon^{-j} f(\epsilon^j z), \quad \epsilon = \exp\left(\frac{2\pi i}{k}\right), \quad k \geq 2,
\]
which holds following relations
\[
f_k(\epsilon^j z) = \epsilon^j f_k(z),
\]
\[
f_k'(\epsilon^j z) = \epsilon^j f'_k(z),
\]
\[
f_k''(\epsilon^j z) = \epsilon^j f''_k(z),
\]
\[f_k'(\epsilon^j z) = \epsilon^j f''_k(z).
\]
For $f = h + g$, where $h$ and $g$ are of the form (2), Jahangiri [4] defined the modified Salagean operator of $f$ as:

$$D^\lambda f(z) = D^\lambda h(z) + (-1)^\lambda D^\lambda g(z); \quad \lambda = 0, 1, 2, \ldots, \quad (3)$$

where

$$D^\lambda h(z) = z^m + (-1)^\lambda \sum_{n=1}^{\infty} n^\lambda a_n z^{-n},$$

$$D^\lambda g(z) = (-1)^\lambda \sum_{n=1}^{\infty} n^\lambda b_n z^{-n}.$$

Involving this operator $D^\lambda$, a class $\sum^*_H k(\lambda, \alpha)$ is defined as follows.

**Definition 1.4** For $\lambda \in N_0$, $0 \leq \alpha < 1$ and $k \geq 2$, let $\sum^*_H k(\lambda, \alpha)$ denote the class of multivalent meromorphic harmonic functions $f$ of the form (2) satisfying

$$\text{Re} \left\{ \frac{D^{\lambda+1} f(z)}{D^\lambda f_k(z)} \right\} > \alpha,$$

where

$$f_k(z) = h_k(z) + g_k(z),$$

$$h_k(z) = z + \sum_{n=1}^{\infty} a_n \psi_n z^{-n} \quad g_k(z) = \sum_{n=1}^{\infty} b_n \psi_n z^{-n}, \quad (5)$$

$$\psi_n = \frac{1}{k} \sum_{j=0}^{k-1} e^{-(n+1)j}, \quad \left( k \geq 2; \quad \epsilon = \exp\left(\frac{2 \pi i}{k}\right)\right) \quad (6)$$

$$= \begin{cases} 1, & n + 1 = lk, \quad l \in N, \\ 0, & n + 1 = lk + 1, \quad l \in N \end{cases}$$

and

$$D^\lambda f_k(z) = D^\lambda h_k(z) + (-1)^\lambda D^\lambda g_k(z).$$

Let $\sum^*_H k(\lambda, \alpha)$ denote the subclass of $\sum^*_H k(\lambda, \alpha)$ consisting of functions of the form $f_\lambda = h_\lambda + \overline{g_\lambda}$ such that

$$h_\lambda(z) = z + (-1)^\lambda \sum_{n=1}^{\infty} |a_n| z^{-n}, \quad g_\lambda(z) = -\sum_{n=1}^{\infty} |b_n| z^{-n}. \quad (7)$$

Also, let $f_{k\lambda} = h_{k\lambda} + \overline{g_{k\lambda}}$ where $h_{k\lambda}$ and $g_{k\lambda}$ are of the form

$$h_{k\lambda}(z) = z + (-1)^\lambda \sum_{n=1}^{\infty} |a_n| \psi_n z^{-n}, \quad g_{k\lambda}(z) = -\sum_{n=1}^{\infty} |b_n| \psi_n z^{-n}, \quad (8)$$

where $\psi_n$ is given by (6).
In this article, a result is based on the class $\sum^*_{H^k}(\lambda, \alpha)$ is obtained and a sufficient coefficient condition for functions $f = h + g$, where $h$ and $g$ are of the form (2) to be in the class $\sum^*_{H^k}(\lambda, \alpha)$ is determined. It is shown that this coefficient condition is also necessary for functions to be in its subclass $\sum^*_{H^k}(\lambda, \alpha)$. Furthermore, bounds and extreme points for functions in $\sum^*_{H^k}(\lambda, \alpha)$ class are obtained.

2 A Result for class $\sum^*_{H^k}(\lambda, \alpha)$

In this section, a result for the class $\sum^*_{H^k}(\lambda, \alpha)$ is derived.

**Theorem 2.1** For $\lambda \in \mathbb{N}_0$, $0 \leq \alpha < 1$ and $k \geq 2$, if $f \in \sum^*_{H^k}(\lambda, \alpha)$, then $D^\lambda f_k(z) \in \sum^*_{H^k}(\lambda, \alpha)$.

**Proof** If $f \in \sum^*_{H^k}(\lambda, \alpha)$, then from the Definition 1.4, it follows that

$$\text{Re} \left\{ \frac{D^{\lambda+1}f(z)}{D^\lambda f_k(z)} \right\} > \alpha, z \in \tilde{U}.$$  

Hence

$$\text{Re} \left\{ \frac{D^{\lambda+1}f(\epsilon^\mu z)}{D^\lambda f_k(\epsilon^\mu z)} \right\} > \alpha, \quad \epsilon^\mu z \in \tilde{U}, \quad \epsilon = \exp\left(\frac{2\pi i}{k}\right), \quad \mu = 0, 1..(k - 1).$$

From the Definition 1.4, it follows that $D^\lambda f_k(\epsilon^\mu z) = \epsilon^\mu D^\lambda f_k(z)$, $\mu = 0, 1..(k - 1)$. Thus

$$\frac{1}{k} \sum_{\mu=0}^{k-1} \text{Re} \left\{ \frac{D^{\lambda+1}f(\epsilon^\mu z)}{D^\lambda f_k(\epsilon^\mu z)} \right\} > \alpha.$$  

Since

$$D^{\lambda+1}f_k(\epsilon^\mu z) = \frac{1}{k} \sum_{j=0}^{k-1} \epsilon^{-j} D^{\lambda+1}f(\epsilon^j z),$$

then

$$\text{Re} \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \epsilon^\mu D^{\lambda+1}f_k(z) \right\} > \alpha$$

or

$$\text{Re} \left\{ \frac{D^{\lambda+1}f_k(z)}{D^\lambda f_k(z)} \right\} > \alpha,$$

which proves the result.
3 Coefficient Conditions

In this section, sufficient coefficient condition for a function $f \in \sum_H$ to be in $\sum^*_H(\lambda, \alpha)$ is derived and then it is shown that this coefficient condition is necessary for its subclass $\sum^*_H(\lambda, \alpha)$.

Theorem 3.1 Let $f = h + g$, where $h$ and $g$ are given by (2) and $f_k = h_k + g_k$, where $h_k$ and $g_k$ are given by (5) satisfying

$$\sum_{n=1}^{\infty} n^{\lambda} \left[ \frac{(n + \alpha \psi_n)}{(1 - \alpha)} |a_n| + \frac{(n - \alpha \psi_n)}{(1 - \alpha)} |b_n| \right] \leq 1,$$

or, equivalently

$$\sum_{l_k=2, k \geq 1}^{\infty} \left[ \frac{l_k - (1 - \alpha)}{(1 - \alpha)} \right] (l_k - 1)^{\lambda} |a_{l_k-1}|$$

$$+ \sum_{l_k=2, k \geq 1}^{\infty} \left[ \frac{l_k - (1 + \alpha)}{(1 - \alpha)} \right] (l_k - 1)^{\lambda} |b_{l_k-1}|$$

$$+ \sum_{l_k=2, k \geq 2}^{\infty} \frac{l_k}{(1 - \alpha)} \left( |a_{l_k}| + |b_{l_k}| \right) \left( \frac{l_k}{(1 - \alpha)} \right)^{\lambda} \leq 1,$$

where $\lambda \in N_0$, $0 \leq \alpha < 1$, $k \in N$, then $f$ is harmonic, orientation-preserving in $\tilde{U}$ and $f \in \sum^*_H(\lambda, \alpha)$.

Proof To show that $f$ is orientation-preserving in $\tilde{U}$, it only needs to show that $|h'(z)| \geq |g'(z)|$ in $\tilde{U}$. Thus

$$|h'(z)| = \left| 1 - \sum_{n=1}^{\infty} n a_n z^{-(n+1)} \right|$$

$$\geq 1 - \sum_{n=1}^{\infty} n |a_n| |z|^{-(n+1)}$$

$$= 1 - \sum_{n=1}^{\infty} n |a_n| r^{-(n+1)}$$

$$\geq 1 - \sum_{n=1}^{\infty} n |a_n|$$

$$\geq 1 - \sum_{n=1}^{\infty} n^{\lambda} \frac{(n + \alpha \psi_n)}{(1 - \alpha)} |a_n|$$

$$\geq \sum_{n=1}^{\infty} n^{\lambda} \frac{(n - \alpha \psi_n)}{(1 - \alpha)} |b_n| \geq \sum_{n=1}^{\infty} n |b_n|$$

$$\geq \sum_{n=1}^{\infty} n |b_n| r^{-(n+1)} \geq \sum_{n=1}^{\infty} n |b_n| |z|^{-(n+1)} \geq |g'(z)|.$$
Now, in order to show \( f \in \sum_h^* (\lambda, \alpha) \), it suffices to show that
\[
\Re \left\{ \frac{D^{\lambda+1} f(z)}{D^\lambda f_k(z)} \right\} > \alpha \tag{10}
\]
or,
\[
\Re \left\{ \frac{D^{\lambda+1} h(z) - (-1)^{\lambda} D^{\lambda+1} g(z)}{D^\lambda h_k(z) + (-1)^{\lambda} D^\lambda g_k(z)} \right\} > \alpha,
\]
where \( z = re^{i\theta} \), \( 0 < r \leq 1 \), \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \alpha < 1 \).

Let
\[
A(z) := D^{\lambda+1} h(z) - (-1)^{\lambda} D^{\lambda+1} g(z) \tag{11}
\]
and
\[
B(z) := D^\lambda h_k(z) + (-1)^{\lambda} D^\lambda g_k(z). \tag{12}
\]
It is observed that (10) holds if
\[
|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \geq 0. \tag{13}
\]
From (11) and (12), it follows that
\[
|A(z) + (1 - \alpha)B(z)| = \left| D^{\lambda+1} h(z) - (-1)^{\lambda} D^{\lambda+1} g(z) + (1 - \alpha) \left( D^\lambda h_k(z) + (-1)^{\lambda} D^\lambda g_k(z) \right) \right|
\]
\[
= \left| (2 - \alpha)z - (-1)^{\lambda} \sum_{n=1}^{\infty} [n - (1 - \alpha)\psi_n] n^\lambda a_n z^{-n} + \sum_{n=1}^{\infty} [n + (1 - \alpha)\psi_n] n^\lambda b_n z^{-n} \right|
\]
\[
\geq (2 - \alpha) |z| - \sum_{n=1}^{\infty} [n - (1 - \alpha)\psi_n] n^\lambda |a_n| |z|^{-n} + \sum_{n=1}^{\infty} [n + (1 - \alpha)\psi_n] n^\lambda |b_n| |z|^{-n}
\]
and
\[
|A(z) - (1 + \alpha)B(z)| = \left| D^{\lambda+1} h(z) - (-1)^{\lambda} D^{\lambda+1} g(z) - (1 + \alpha) \left( D^\lambda h_k(z) + (-1)^{\lambda} D^\lambda g_k(z) \right) \right|
\]
\[
= \left| (-\alpha)z - (-1)^{\lambda} \sum_{n=1}^{\infty} [n + (1 + \alpha)\psi_n] n^\lambda a_n z^{-n} + \sum_{n=1}^{\infty} [n - (1 + \alpha)\psi_n] n^\lambda b_n z^{-n} \right|
\]
\[
= |\alpha z + (-1)^{\lambda} \sum_{n=1}^{\infty} [n + (1 + \alpha)\psi_n] n^\lambda a_n z^{-n} - \sum_{n=1}^{\infty} [n - (1 + \alpha)\psi_n] n^\lambda b_n z^{-n}|
\]
\[
\leq \alpha z + \sum_{n=1}^{\infty} [n + (1 + \alpha)\psi_n] n^\lambda |a_n| |z|^{-n} - \sum_{n=1}^{\infty} [n - (1 + \alpha)\psi_n] n^\lambda |b_n| z^{-n}
\]
Thus
\[
|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)|
\]
\[
\geq 2(1 - \alpha) |z| - 2\sum_{n=1}^{\infty} [n + \alpha \psi_n] n^\lambda |a_n| |z|^{-n}
- 2\sum_{n=1}^{\infty} [n - \alpha \psi_n] n^\lambda |b_n| |z|^{-n}
\]

\[
\geq 2 |z| \left\{ (1 - \alpha) - \sum_{n=1}^{\infty} \frac{[n + \alpha \psi_n]}{(1 - \alpha)} n^\lambda |a_n| |z|^{-(n+1)}
- \sum_{n=1}^{\infty} \frac{[n - \alpha \psi_n]}{(1 - \alpha)} n^\lambda |b_n| |z|^{-(n+1)} \right\}
\]

\[
\geq 2 |z| (1 - \alpha) \left\{ 1 - \sum_{n=1}^{\infty} \frac{[n + \alpha \psi_n]}{(1 - \alpha)} n^\lambda |a_n| |z|^{-(n+1)}
- \sum_{n=1}^{\infty} \frac{[n - \alpha \psi_n]}{(1 - \alpha)} n^\lambda |b_n| |z|^{-(n+1)} \right\}
\]

\[
\geq 2(1 - \alpha) \left\{ 1 - \sum_{n=1}^{\infty} \frac{[n + \alpha \psi_n]}{(1 - \alpha)} n^\lambda |a_n|
+ \sum_{n=1}^{\infty} \frac{[n - \alpha \psi_n]}{(1 - \alpha)} n^\lambda |b_n| \right\}
\] (14)

From the definition of \( \psi_n \), it follows that

\[
\psi_n = \begin{cases} 
1, & n + 1 = lk, l \in \mathbb{N}, k \geq 1 \\
0, & n + 1 = lk + 1, l \in \mathbb{N}, k \geq 2.
\end{cases}
\] (15)

Substituting (15) in (14), then (14) is equivalent to

\[
|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)|
\geq 2 (1 - \alpha) \left\{ 1 - \sum_{lk=2, k \geq 1} \frac{[lk - (1 - \alpha)]}{(1 - \alpha)} (lk - 1)^\lambda |a_{lk-1}|
- \sum_{lk=2, k \geq 1} \frac{[lk - (1 + \alpha)]}{(1 - \alpha)} \left( \frac{lk - m}{m} \right)^\lambda |b_{lk-m}|
- \sum_{lk=2, k \geq 2} \left( \frac{lk}{(1 - \alpha)} \right) (|a_{lk}| + |b_{lk}|) (lk)^\lambda \right\}
\]

\[
\geq 0. \text{ by (13).}
\]

Thus, this completes the proof of the Theorem.

**Theorem 3.2** Let \( f_\lambda = h_\lambda + \overline{g_\lambda} \), where \( h_\lambda \) and \( g_\lambda \) are of the form (7), and \( f_{k\lambda} = h_{k\lambda} + \overline{g_{k\lambda}} \), where \( h_{k\lambda} \) and \( g_{k\lambda} \) are of the form (8). Then, \( f_\lambda \in \sum_{\lambda}^*(\lambda, \alpha) \), if and only if inequality (9) holds for the coefficient of \( f_\lambda = h_\lambda + \overline{g_\lambda} \) and \( f_{k\lambda} = h_{k\lambda} + \overline{g_{k\lambda}} \).
Proof Since $\sum_{H^\alpha}(\lambda, \alpha) \subset \sum_{H_k}(\lambda, \alpha)$, it only needs to prove the “only if” part of the Theorem. For this, it suffices to show that $f_\lambda \notin \sum_{H^\alpha}(\lambda, \alpha)$ if the condition (9) does not hold. If $f_\lambda \in \sum_{H^\alpha}(\lambda, \alpha)$, then writing corresponding series expansions in (4), it follows that $\Re \left\{ \frac{\xi(z)}{\eta(z)} \right\} \geq 0$ for all values of $z$ in $\bar{U}$, where

$$
\xi(z) = D^{\lambda+1}h_\lambda(z) - (-1)^\lambda D^{\lambda+1}g_\lambda(z) - \alpha D^\lambda h_k(z) - \alpha(-1)^\lambda D^\lambda g_k(z)
$$

$$
= z - \sum_{n=1}^{\infty} \left[ \frac{n + \alpha \psi_n}{(1 - \alpha)} \right] (n)^\lambda |a_n| z^{-n} - \sum_{n=1}^{\infty} \left[ \frac{n - \alpha \psi_n}{(1 - \alpha)} \right] n^\lambda |b_n| z^{-n}
$$

and

$$
\eta(z) = D^\lambda h_k(z) + (-1)^\lambda D^\lambda g_k(z)
$$

$$
= z + \sum_{n=1}^{\infty} n^\lambda \psi_n \left[ |a_n| z^{-n} + |b_n| z^{-n} \right].
$$

Since

$$
\left| \frac{\xi(z)}{\eta(z)} \right| \geq \Re \left\{ \frac{\xi(z)}{\eta(z)} \right\} \geq 0,
$$

hence the condition $\Re \left\{ \frac{\xi(z)}{\eta(z)} \right\} \geq 0$ is equivalent to

$$
1 - \sum_{n=1}^{\infty} n^\lambda \left[ \frac{n + \alpha \psi_n}{(1 - \alpha)} \right] |a_n| + \sum_{n=1}^{\infty} n^\lambda \psi_n \left[ |a_n| + |b_n| \right] r^{-(n+1)} \geq 0.
$$

(16)

Now if the condition (9) does not holds then the numerator of (16) is non-positive for $r$ sufficiently close to 1, which contradicts that $f_\lambda \in \sum_{H^\alpha}(\lambda, \alpha)$ and this proves the required result.

Taking $\lambda = 0$, in Theorems 3.1 and 3.2, following results are obtained.

Corollary 3.3 Let $f = h + g$, where $h$ and $g$ are given by (2) and $f_k = h_k + g_k$, where $h_k$ and $g_k$ are given by (3) satisfying

$$
\sum_{n=1}^{\infty} \left[ \frac{n + \alpha \psi_n}{(1 - \alpha)} \right] |a_n| + \sum_{n=1}^{\infty} \left[ \frac{n - \alpha \psi_n}{(1 - \alpha)} \right] |b_n| \leq 1,
$$

or, equivalently

$$
\sum_{l_k=2, k \geq 1}^{\infty} \left[ \frac{l_k - (1 - \alpha)}{(1 - \alpha)} \right] |a_{l_k-1}| + \sum_{l_k=2, k \geq 1}^{\infty} \left[ \frac{l_k - (1 + \alpha)}{(1 - \alpha)} \right] |b_{l_k-1}|
$$

$$
+ \sum_{l_k=2, k \geq 2} \left( \frac{l_k}{(1 - \alpha)} \right) (|a_{l_k}| + |b_{l_k}|) \leq 1,
$$

where $0 \leq \alpha < 1$, $k \in N$, then $f$ is harmonic, orientation-preserving in $\bar{U}$ and $f \in \sum_{H^\alpha}(\lambda, \alpha)$. Furthermore, $f \in \sum_{H^\alpha}(\lambda, \alpha)$, if and only if above inequality holds.
4 Bounds and Extreme Points

In this section, bounds and extreme points for functions belonging to the class \( \mathcal{H}^\ast_k(\lambda, \alpha) \) are estimated.

**Theorem 4.1** If \( f_\lambda = h_\lambda + g_\lambda \in \mathcal{H}^\ast_k(\lambda, \alpha) \) for \( 0 \leq \alpha < 1, 0 < |z| = r < 1 \), then

\[
|f_\lambda(z)| \leq |z + (-1)\lambda \sum_{n=1}^{\infty} |a_n| z^{-n} - \sum_{n=1}^{\infty} |b_n| z^{-n}|
\]

**Proof** Let \( f_\lambda = h_\lambda + g_\lambda \in \mathcal{H}^\ast_k(\lambda, \alpha) \). Taking the absolute value of \( f_\lambda \) it follows that

\[
|f_\lambda(z)| = \left| z + (-1)\lambda \sum_{n=1}^{\infty} |a_n| z^{-n} - \sum_{n=1}^{\infty} |b_n| z^{-n} \right|
\]

\[
\leq r + \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^{-n}
\]

\[
\leq r + r^{-1} \sum_{n=1}^{\infty} (|a_n| + |b_n|)
\]

\[
\leq r + r^{-1} \sum_{n=1}^{\infty} \left( (1 - \alpha) \frac{n - \alpha \psi_n}{1 - \alpha} \frac{1}{n^\lambda} \right) (|a_n| + |b_n|)
\]

\[
\leq r + r^{-1} \sum_{n=1}^{\infty} \left( n + \alpha \psi_n \frac{1}{(1 - \alpha)} |a_n| + n - \alpha \psi_n \frac{1}{(1 - \alpha)} |b_n| \right)
\]

\[
\leq r + r^{-1} \sum_{n=1}^{\infty} \left( |a_n| + |b_n| \right)
\]

and

\[
|f_\lambda(z)| = \left| z + (-1)\lambda \sum_{n=1}^{\infty} |a_n| z^{-n} - \sum_{n=1}^{\infty} |b_n| z^{-n} \right|
\]

\[
\geq r - \sum_{n=1}^{\infty} (|a_n| + |b_n|) r^{-n}
\]

\[
\geq r - r^{-1} \sum_{n=1}^{\infty} (|a_n| + |b_n|)
\]

\[
\geq r - r^{-1} \sum_{n=1}^{\infty} \left( (1 - \alpha) \frac{n - \alpha \psi_n}{1 - \alpha} \frac{1}{n^\lambda} \right) (|a_n| + |b_n|)
\]

\[
\geq r - r^{-1} \sum_{n=1}^{\infty} \left( n + \alpha \psi_n \frac{1}{(1 - \alpha)} |a_n| + n - \alpha \psi_n \frac{1}{(1 - \alpha)} |b_n| \right)
\]

\[
\geq r - r^{-1} \sum_{n=1}^{\infty} \left( |a_n| + |b_n| \right)
\]

\[
\geq r - r^{-1}.
\]
This proves the required result.

The bounds given in Theorem 4.1 holds for the functions \( f_\lambda = h_\lambda + g_\lambda \), and it also found that these bounds also holds for functions \( f = h + g \).

**Theorem 4.2** Let \( f_\lambda = h_\lambda + g_\lambda \), where \( h_\lambda \) and \( g_\lambda \) are of the form (7) then \( f_\lambda \in \sum_{\mathcal{H}^k}(\lambda, \alpha) \), if and only if \( f_\lambda \) can be expressed as:

\[
f_\lambda(z) = \sum_{n=0}^{\infty} (x_n h_\lambda_n(z) + y_n g_\lambda_n(z)),
\]

where \( z \in \tilde{U} \) and

\[
h_\lambda_0(z) = z, \quad h_\lambda_n(z) = z + (-1)^\lambda \frac{(1-\alpha)}{[n + \alpha \psi_n] n^{\lambda}} z^{-n},
\]

\[
g_\lambda_0(z) = z, \quad g_\lambda_n(z) = z - \frac{(1-\alpha)}{[n - \alpha \psi_n] n^{\lambda}} z^{-n}
\]

for \( n = 1, 2, 3, \ldots \), and

\[
\sum_{n=0}^{\infty} (x_n + y_n) = 1, \quad x_n, y_n \geq 0.
\]

In particular, the extreme points of \( \sum_{\mathcal{H}^k}(\lambda, \alpha) \) are \( h_\lambda_n \) and \( g_\lambda_n \).

**Proof** Let

\[
f_\lambda(z) = \sum_{n=0}^{\infty} (x_n h_\lambda_n(z) + y_n g_\lambda_n(z))
\]

\[
= x_0 h_\lambda_0(z) + y_0 g_\lambda_0(z) + \sum_{n=1}^{\infty} x_n \left( z + (-1)^\lambda \frac{(1-\alpha)}{[n + \alpha \psi_n] n^{\lambda}} z^{-n} \right)
\]

\[
\quad + \sum_{n=1}^{\infty} y_n \left( z - \frac{(1-\alpha)}{[n - \alpha \psi_n] n^{\lambda}} z^{-n} \right)
\]

\[
= \sum_{n=0}^{\infty} (x_n + y_n) z + \sum_{n=1}^{\infty} \left\{ (-1)^\lambda \left( \frac{(1-\alpha)}{[n + \alpha \psi_n] n^{\lambda}} \right) x_n z^{-n} - \frac{(1-\alpha)}{[n - \alpha \psi_n] n^{\lambda}} y_n z^{-n} \right\}.
\]

Thus by Theorem 3.2, it is noted that \( f_\lambda \in \sum_{\mathcal{H}^k}(\lambda, \alpha) \), since,

\[
\sum_{n=1}^{\infty} n^{\lambda} \left\{ \frac{[n + \alpha \psi_n]}{(1-\alpha)} \left( \frac{(1-\alpha)}{[n + \alpha \psi_n] n^{\lambda}} x_n \right) \right. \\
\quad + \left. \frac{[n - \alpha \psi_n]}{1-\alpha} \left( \frac{(1-\alpha)}{[n - \alpha \psi_n] n^{\lambda}} y_n \right) \right\}
\]

\[
= \sum_{n=1}^{\infty} (x_n + y_n) = (1 - x_0 - y_0) \leq 1.
\]
Conversely, suppose that \( f_\lambda \in \sum_{H^k(\lambda, \alpha)} \), then (9) holds. Setting

\[
\begin{align*}
x_n &= \frac{[n + \alpha\psi_n]}{(1 - \alpha)} n^\lambda |a_n|, \\
y_n &= \frac{[n - \alpha\psi_n]}{(1 - \alpha)} n^\lambda |b_n|, 
\end{align*}
\]

which satisfy (14), thus

\[
f_\lambda(z) = z + \sum_{n=1}^{\infty} \frac{(1 - \alpha)}{n[a_n] + \alpha\psi_n} x_n z^{-n} - \sum_{n=1}^{\infty} \frac{(1 - \alpha)}{n[a_n] - \alpha\psi_n} y_n z^{-n}
\]

\[
= z + \sum_{n=1}^{\infty} [h_{\lambda_n} - z] x_n + \sum_{n=1}^{\infty} [g_{\lambda_n} - z] y_n
\]

\[
= z \left[ 1 - \sum_{n=1}^{\infty} x_n - \sum_{n=1}^{\infty} y_n \right] + \sum_{n=1}^{\infty} h_{\lambda_n} x_n + \sum_{n=1}^{\infty} g_{\lambda_n} y_n
\]

\[
= x_0 h_{\lambda_0} + y_0 g_{\lambda_0} + \sum_{n=1}^{\infty} h_{\lambda_n} x_n + \sum_{n=1}^{\infty} g_{\lambda_n} y_n
\]

\[
= \sum_{n=0}^{\infty} (x_n h_{\lambda_n}(z) + y_n g_{\lambda_n}(z)).
\]

This proves the Theorem.

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**References**


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