Certain Subclasses of Multivalent Functions
Associated with Fractional Calculus Operator

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Abstract. By means of the generalized fractional calculus operator, we introduce and investigate a subclass of \( p \)-valent functions with negative coefficients. We obtain the coefficient estimates, distortion bounds for the class. Furthermore, we discuss inclusion relationship involving the neighborhood result of the same.

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1. INTRODUCTION

Let \( A_p(n) \) denote the class of functions of the form

\[
(1.1.1) \quad f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, 3, \ldots \}),
\]

which are analytic and \( p \)-valent in the open unit disk \( U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\} \).
Let \( T_p(n) \) be the subclass of \( A_p(n) \), consisting of functions of the form

\[
(1.1.2) \quad f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N}; a_k \geq 0)
\]

which are \( p \)-valent in \( U \).

The Hadamard product of two power series

\[
f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k
\]

is defined by

\[
(1.1.3) \quad (f * g)(z) = z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k.
\]

Let \( \phi_p(a, c; z) \) be the incomplete beta function defined by

\[
(1.1.4) \quad \phi_p(a, c; z) = z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k, \quad a \in \mathbb{R}, \quad c \in \mathbb{R} \setminus \mathbb{Z}^- = \{\ldots, -2, -1, 0\}, \quad z \in U,
\]

where \((a)_k\) is the Pochhammer symbol defined by

\[
(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = \begin{cases} 1, \quad k = 0 \\ a(a+1)(a+2)\ldots(a+k-1), \quad k \in \mathbb{N}. \end{cases}
\]

Further, for \( f \in A_p(n) \)

\[
L_p(a, c) f(z) = \phi_p(a, c; z) * f(z),
\]

\[
= z^p + \sum_{k=n+p}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} a_k z^k, \quad z \in U,
\]

where \( L_p(a, c) \) leads to Saitoh operator [22] for \( n = 1 \) and yields Carlson-Shaffer operator [4] for \( n = 1 \) and \( p = 1 \).

**Definition 1.1.** For real numbers \( \mu > 0, \gamma \) and \( \eta \), Saigo hypergeometric fractional integral operator \( I_{0, z}^{\mu, \gamma, \eta} \) is defined by

\[
(1.1.5) \quad I_{0, z}^{\mu, \gamma, \eta} f(z) = \frac{z^{-\mu-\gamma}}{\Gamma(\mu)} \int_{0}^{z} (z-t)^{\mu-1} F_1 \left( \mu + \gamma, -\eta; \mu; 1 - \frac{t}{z} \right) f(t) dt,
\]

where the function \( f(z) \) is analytic in a simply-connected region of the complex \( z \)-plane containing the origin, with the order

\[
f(z) = O(|z|^\xi) \quad (z \to 0; \xi > \max\{0, \gamma - \eta\} - 1)
\]

and the multiplicity of \((z-t)^{\mu-1}\) is removed by requiring \( \log(z-t) \) to be real when \((z-t) > 0\).
**Definition 1.2.** Under the hypotheses of Definition 1.1, Saigo hypergeometric fractional derivative operator $J_{0,z}^{\mu,\gamma,\eta}$ is defined by

\[
J_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} 
\left\{ \frac{d}{dt} \left[ z^{\mu-\gamma} \int_0^{z-t)^{-\mu} F_1(\gamma-\mu,1-\eta;1-\mu;1-\frac{1}{z}) f(t) dt \right] \right\}, & 0 \leq \mu < 1, \\
\frac{d^k}{dz^k} J_{0,z}^{\mu-k,\gamma,\eta} f(z), & k \leq \mu < k + 1; k \in \mathbb{N},
\end{cases}
\]

where the multiplicity of $z-t)^{-\mu}$ is removed as in Definition 1.1.

It may be remarked that

\[
I_{0,z}^{\mu,\gamma,\eta} f(z) = D_z^{-\mu} f(z) \quad (\mu > 0)
\]

and

\[
J_{0,z}^{\mu,\gamma,\eta} f(z) = D_z^\mu f(z) \quad (0 \leq \mu < 1),
\]

where $D_z^{-\mu}$ denotes fractional integral operator and $D_z^\mu$ denotes fractional derivative operator considered by Owa [15] and subsequently by Owa and Srivastava [16].

Let $U_{0,z}^{\mu,\gamma,\eta} : A_p(n) \rightarrow A_p(n)$, be a generalized fractional differintegral operator defined by

\[
U_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} 
\frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu-\gamma)} z^\gamma J_{0,z}^{\mu,\gamma,\eta} f(z), & 0 \leq \mu < \eta + p + 1, z \in \mathbb{U}, \\
\frac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu-\gamma)} z^\gamma I_{0,z}^{\mu,\gamma,\eta} f(z), & -\infty < \mu < 0, z \in \mathbb{U}.
\end{cases}
\]

For $n = 1$ the generalized fractional differintegral operator was considered by Goyal and Prajapat [7] (see also [18]). It is easily seen from (1.1.7) that for a function $f$ of the form (1.1.1), we have

\[
U_{0,z}^{\mu,\gamma,\eta} f(z) = z^p + \sum_{k=n+p}^{\infty} \frac{(1+p)_k(1+p+\eta-\gamma)_k}{(1+p-\gamma)_k(1+p+\eta-\mu)_k} a_k z^k
\]

\[
(1.1.8)
\]

\[
z^p F_1(1,1+p,1+p+\eta-\gamma;1+p-\gamma,1+p+\eta-\mu; z) \ast f(z)
\]

\[
(z \in \mathbb{U}; p \in \mathbb{N}; \gamma, \eta \in \mathbb{R}; \gamma < p + 1; -\infty < \mu < \eta + p + 1),
\]

where $\ast$ is given by (1.1.3) and $p F_q$ is well known generalized hypergeometric function.

Note that

\[
U_{0,z}^{0,0,0} f(z) = f(z), \quad U_{0,z}^{1,1,1} f(z) = U_{0,z}^{1,0,0} f(z) = \frac{zf'(z)}{p} \text{ and } U_{0,z}^{2,1,1} f(z) = \frac{zf'(z) + z^2 f''(z)}{p^2}.
\]

We also remark that

\[
U_{0,z}^{\mu,\mu,\eta} f(z) = U_{0,z}^{\mu,\gamma,0} f(z) = \Omega_{0,z}^{\mu,\mu} f(z),
\]

where $\Omega_{0,z}^{\mu,\mu}$ is an extended fractional differintegral operator studied very recently by [17] (see also [19]). On the other hand, if we set

\[
\mu = -\delta, \quad \gamma = 0 \text{ and } \eta = \nu - 1,
\]
in (1.1.8) and using

\[ I_{0,z}^{\delta,\nu-1} f(z) = \frac{1}{z^\nu \Gamma(\delta)} \int_0^z \frac{t^{\nu-1}}{1 - \frac{t}{z}} f(t) dt, \]

we obtain following \( p \)-valent generalization of multiplier transformation operator [9, 10]:

\[ (Q_\delta^{\nu,p}) f(z) = \left(\frac{p + \delta + \nu - 1}{p + \nu - 1}\right) \frac{\delta}{z^\nu} \int_0^z \frac{t^{\nu-1}}{1 - \frac{t}{z}} f(t) dt \]

\[ = z^p + \sum_{k=n+p}^{\infty} \frac{\Gamma(\nu + k) \Gamma(p + \delta + \nu)}{\Gamma(\delta + \nu + k) \Gamma(p + \nu)} a_k z^k \quad (\nu > -p, \delta + \nu > -p) \]

on the other hand, if we set

\[ \mu = -1, \quad \gamma = 0 \quad \text{and} \quad \eta = \nu - 1, \]

in (1.1.8), we obtain the generalized Berardi-Libera integral operator \( F_{\nu,p} : A_p(n) \to A_p(n) \ (\nu > -p) \) defined by

\[
F_{\nu,p} f(z) = \frac{p + \nu}{z^\nu} \int_0^z \frac{t^{\nu-1}}{1 - \frac{t}{z}} f(t) dt
\]

\[ = z^p + \sum_{k=n+p}^{\infty} \frac{p + \nu}{\nu + k} a_k z^k \quad (\nu > -p). \]

For the choice \( p = 1, n = 1 \) and \( \nu \in \mathbb{N} \), the operator defined by (1.1.10) reduces to the well-known Bernardi integral operator. Also the operators \( Q_\delta^{\nu,p} \) and \( F_{\nu,p} \) were studied recently in [12].

Further, For \( f \in A_p(n) \)

\[ M_{\alpha,z}^{\mu,\gamma,\eta} f(z) = \phi_p(a, c, z) \ast U_{\alpha,z}^{\mu,\gamma,\eta} f(z) \]

\[ = z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}} a_k z^k \]

\[ = z^p + \sum_{k=p+n}^{\infty} \Phi(k) a_k z^k \quad (z \in \mathbb{U}), \]

where,

\[ \Phi(k) = \frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}}. \]

The operator \( M_{\alpha,z}^{\mu,\gamma,\eta} f(z) \) was studied by Kahairnar and More [8].
**Definition 1.3.** A function $f \in T_p(n)$ is said to be in the class $S(b, \lambda; p)$ if it satisfies the following inequality

$\left| \frac{1}{b} \left( \frac{z(M_{0,z}^{\mu,\gamma,n} f(z))'}{(1 - \lambda)M_{0,z}^{\mu,\gamma,n} f(z)} + \lambda z^2 (M_{0,z}^{\mu,\gamma,n} f(z))'' - p \right) \right| < \beta \quad (z \in \mathbb{U}),$

where $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, p \in \mathbb{N}, 0 \leq \lambda \leq 1, 0 < \beta \leq 1.$

Our definition of the function class $S(b, \lambda; p)$ is motivated essentially by the earlier investigations of Aouf [3], Darwich and Aouf [5], Mahzoon and Latha [11], Orhan and Kamali [14], Raina and Srivastava [20], in each of which further details and references to other closely related subfamilies of the normalized $p$-valently analytic function class $T_p(n)$ can be found.

The object of the present paper is to investigate the various properties and characteristics of analytic $p$-valent functions belonging to the subclass $S(b, \lambda; p)$ which we have defined here. Apart from deriving coefficient bounds for the function class, we establish distortion bounds, inclusion relationship involving the $(n, \delta)$—neighborhoods of analytic $p$—valent functions belonging to this subclass.

### 2. Coefficient Estimates

In this section, we determine the coefficient inequality for functions to be in the subclass $S(b, \lambda; p)$.

**Theorem 2.1.** A function $f(z)$ defined by (1.1.2) is in $S(b, \lambda; p)$ if and only if

$$\sum_{k=p+n}^{\infty} [1 + \lambda(k - 1)](k - p + \beta |b|)\Phi(k)a_k \leq \beta |b|[1 + \lambda(p - 1)],$$

where $0 \leq \lambda \leq 1, 0 < \beta \leq 1$ and $\Phi(k)$ is given by (1.1.12).

**Proof.** Let $f(z) \in S(b, \lambda; p)$. Assume that inequality (2.2.1) holds true. Then we find that

$$\frac{z(M_{0,z}^{\mu,\gamma,n} f(z))'}{(1 - \lambda)M_{0,z}^{\mu,\gamma,n} f(z)} + \lambda z^2 (M_{0,z}^{\mu,\gamma,n} f(z))'' - p \leq \sum_{k=p+n}^{\infty} [1 + \lambda(k - 1)](k - p + \beta |b|)\Phi(k)a_k z^k$$

$$\leq \frac{\sum_{k=p+n}^{\infty} [1 + \lambda(k - 1)](k - p)\Phi(k)a_k z^k}{[1 + \lambda(p - 1)] + \sum_{k=p+n}^{\infty} [1 + \lambda(k - 1)]\Phi(k)a_k z^k} < \beta |b|.$$
Choosing values of $z$ on real axis and letting $z \to 1^-$, we have
\[
\sum_{k=p+n}^{\infty} \left[1 + \lambda(k-1)\right](k - p + \beta|b|)\Phi(k) a_k \leq \beta|b|\left[1 + \lambda(p - 1)\right].
\]

Conversely, assume that $f(z) \in S(b, \lambda, \beta; p)$, then in the view of (1.1.11) and (1.1.13), we get the following inequality
\[
Re \left\{ \frac{z(M^{n+\eta}_{0,z} f(z))'}{(1 - \lambda)M^{n+\eta}_{0,z} f(z) + \lambda z(M^{n+\eta}_{0,z} f(z))'} - p \right\} > -\beta|b|
\]
\[
Re \left\{ \frac{pz^p[1 + \lambda(p - 1)] + \sum_{k=p+1}^{\infty} k[1 + \lambda(k-1)]\Phi(k) a_k z^k}{[1 + \lambda(p - 1)]z^p + \sum_{k=p+1}^{\infty} [1 + \lambda(k-1)]\Phi(k) a_k z^k} - p + \beta|b| \right\} > 0
\]
\[
Re \left\{ \frac{\beta|b|[1 + \lambda(p - 1)]z^p + \sum_{k=p+1}^{\infty} [1 + \lambda(k-1)](k - p + \beta|b|)\Phi(k) a_k z^k}{[1 + \lambda(p - 1)]z^p + \sum_{k=p+1}^{\infty} [1 + \lambda(k-1)]\Phi(k) a_k z^k} \right\} > 0.
\]

Since $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the above inequality reduces to
\[
\frac{\beta|b|[1 + \lambda(p - 1)]r^p - \sum_{k=p+1}^{\infty} [1 + \lambda(k-1)](k - p + \beta|b|)\Phi(k) a_k r^k}{[1 + \lambda(p - 1)]r^p - \sum_{k=p+1}^{\infty} [1 + \lambda(k-1)]\Phi(k) a_k r^k} > 0.
\]

Letting $r \to 1^-$ and by the mean value theorem we have desired inequality (2.2.1). This completes the proof of the Theorem 2.1.

The result is sharp for the function $f(z)$ given by
\[
f(z) = z^p + \frac{\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(k-1)](k - p + \beta|b|)\Phi(k)} z^k.
\]

**Corollary 2.1.** Let the function $f$ defined by (1.1.2) be in the class $S(b, \lambda, \beta; p)$. Then
\[
a_k \leq \frac{\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(k-1)](k - p + \beta|b|)\Phi(k)} (k \geq p + n, n \in N),
\]
where $\Phi(k)$ is given by (1.1.12).

3. Distortion Bounds

In this section, we obtain distortion bounds for functions in the class $S(b, \lambda, \beta; p)$.

**Theorem 3.1.** Let the function $f$ defined by (1.1.2) be in the class $S(b, \lambda, \beta; p)$. Then for $|z| = r$ we have
\[
r^p - \frac{\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)](n + \beta|b|)\Phi(p + n)} r^{p+n} \leq |f(z)|
\]
\[
\leq r^p + \frac{\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)](n + \beta|b|)\Phi(p + n)} r^{p+n}
\]
and the result are sharp for \( f \) given by

\[
f(z) = z^p + \frac{\beta |b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta|b|\Phi(p + n)]} z^{p+n}.
\]

**Proof.** Given that \( f(z) \in S(b, \lambda, \beta; p) \), from the equation (2.2.1), we have

\[
[1 + \lambda(p + n - 1)][n + \beta|b|\Phi(p + n)] \sum_{k=p+n}^{\infty} a_k \\
\leq \sum_{k=p+n}^{\infty} [1 + \lambda(k - 1)](k - p + \beta|b|) \Phi(k) a_k \leq \beta|b|[1 + \lambda(p - 1)],
\]

which is equivalent to,

\[
(3.3.1) \sum_{k=p+n}^{\infty} a_k \leq \frac{\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta|b|\Phi(p + n)]}.
\]

Using (1.1.2) and (3.3.1), we obtain

\[
f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \\
|f(z)| \leq |z|^p + \sum_{k=p+n}^{\infty} a_k |z|^k \\
\leq \sum_{k=p+n}^{\infty} a_k r^k \\
\leq \sum_{k=p+n}^{\infty} \frac{\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta|b|\Phi(p + n)]} r^{p+n}.
\]

Similarly,

\[
f(z) \geq r^p - \sum_{k=p+n}^{\infty} \frac{\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta|b|\Phi(p + n)]} r^{p+n}.
\]

This completes the proof of Theorem 3.1. \( \square \)

**Theorem 3.2.** Let the function \( f \) defined by (1.1.2) be in the class \( S(b, \lambda, \beta; p) \). Then for \( |z| = r \) we have

\[
pr^{p-1} - \frac{(p + n)\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta|b|\Phi(p + n)]} r^{p+n-1} \\
\leq |f'(z)| \leq pr^{p-1} + \frac{(p + n)\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta|b|\Phi(p + n)]} r^{p+n-1}
\]
and the result are sharp for \( f \) given by
\[
f(z) = z^p + \frac{\beta |b| [1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta |b| \Phi(p + n)]} z^{p+n}.\]

Proof. From (1.1.2) and (3.3.1)
\[
f'(z) = pz^{p-1} + \sum_{k=p+n}^{\infty} k a_k z^{k-1}
\]
\[
|f'(z)| \leq p|z|^{p-1} + \sum_{k=p+n}^{\infty} k a_k |z|^{k-1}
\]
\[
\leq pr^{p-1} + \sum_{k=p+n}^{\infty} k a_k r^{k-1}
\]
\[
\leq pr^{p-1} + \frac{(p+n)\beta |b| [1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta |b| \Phi(p + n)]} r^{p+n-1}.
\]

Similarly,
\[
f'(z) \geq pr^{p-1} - \frac{(p+n)\beta |b| [1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta |b| \Phi(p + n)]} r^{p+n-1}.
\]

This completes the proof of Theorem 3.2. \( \square \)

4. Inclusion and Neighborhood Results

In this section, we prove certain inclusion relationship for functions belonging to the class \( S(b, \lambda, \beta; p) \) and also, we determine the neighborhood properties of functions belonging to the subclass \( S^\alpha(b, \lambda, \beta; p) \).

Following the works of Goodman [6], Ruscheweyh [21] and Altintas et al. [1, 2] (see also [13, 14, 20]) we defined the \((n, \delta)-\) neighborhood of a function \( f \in T_p(n) \) by

\[
N_{n,\delta}(f) = \left\{ g \in T_p(n) : g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \text{ and } \sum_{k=p+n}^{\infty} k |a_k - b_k| \leq \delta \right\}.
\]

In particular, for the function \( e(z) = z^p \) \( (p \in \mathbb{N}) \)

\[
N_{n,\delta}(e) = \left\{ g \in T_p(n), g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \text{ and } \sum_{k=p+n}^{\infty} k |b_k| \leq \delta \right\}.
\]

A function \( f \in T_p(n) \) defined by (1.1.2) is said to be in the class \( S^\alpha(b, \lambda, \beta; p) \) if there exists a function \( h \in S(b, \lambda, \beta; p) \), such that

\[
\left| \frac{f(z)}{h(z)} - 1 \right| < p - \alpha \quad (z \in \mathbb{U}, 0 \leq \alpha < p).
\]
Theorem 4.1. If $\Phi(k) \geq \Phi(p + n)$ for $k \geq p + n,$ $n \in \mathbb{N}$ and
\[
\delta = \frac{(p + n)\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta|b|]\Phi(p + n)}
\]
then
\[
S(b, \lambda, \beta; p) \subseteq N_{n, \delta}(e).
\]
\[\text{Proof.}\] Let $f \in S(b, \lambda, \beta; p).$ Then in view of assertion (2.2.1) of Theorem 2.1
and the condition $\Phi(k) \geq \Phi(p + n)$ for $k \geq p + n,$ $n \in \mathbb{N},$ we get
\[
[1 + \lambda(p + n - 1)][n + \beta|b|]\Phi(p + n) \sum_{k=p+n}^{\infty} a_k \leq \sum_{k=p+n}^{\infty} [1 + \lambda(k - 1)](k - p + \beta|b|)\Phi(k)a_k
\]
(4.4.6)
\[
\leq \beta|b|[1 + \lambda(p - 1)],
\]
which implies
\[
\sum_{k=p+n}^{\infty} a_k \leq \frac{\beta|b|[1 + \lambda(p - 1)]}{[1 + \lambda(p + n - 1)][n + \beta|b|]\Phi(p + n)}.
\]
Applying assertion (2.2.1) of Theorem (2.1) in conjunction with (4.4.7), we obtain
\[
[1 + \lambda(p + n - 1)][n + \beta|b|]\Phi(p + n) \sum_{k=p+n}^{\infty} a_k \leq \beta|b|[1 + \lambda(p - 1)]
\]
\[
(p + n)[1 + \lambda(p + n - 1)][n + \beta|b|]\Phi(p + n) \sum_{k=p+n}^{\infty} a_k \leq (p + n)\beta|b|[1 + \lambda(p - 1)]
\]
\[
\sum_{k=p+n}^{\infty} k|a_k - b_k| \leq \delta,
\]
which by virtue of (4.4.1) establishes the inclusion relation (4.4.5).

Theorem 4.2. If $h \in S(b, \lambda, \beta; p)$ and
\[
\alpha = p - \frac{\delta}{(p + n)} \left[\frac{[1 + \lambda(p + n - 1)][n + \beta|b|]\Phi(p + n)}{[1 + \lambda(p + n - 1)][n + \beta|b|]\Phi(p + n) - \beta|b|[1 + \lambda(p - 1)]}\right],
\]
then $N_{n, \delta}(h) \subset S^\alpha(b, \lambda, \beta; p).$
\[\text{Proof.}\] Suppose that $f \in N_{n, \delta}(h),$ we then find from (4.4.1) that
\[
\sum_{k=p+n}^{\infty} k|a_k - b_k| \leq \delta,
which readily implies the following co-efficient inequality

\begin{equation}
\sum_{k=p+n}^{\infty} |a_k - b_k| \leq \frac{\delta}{p+n} \quad (n \in \mathbb{N}).
\end{equation}

(4.4.9)

Next, since \( h \in S(b, \lambda, \beta; p) \) in the view of (4.4.7), we have

\begin{equation}
\sum_{k=p+n}^{\infty} b_k \leq \frac{\beta |b|[1 + \lambda(p-1)]}{[1 + \lambda(p + n - 1)][1 + \beta|b|]\Phi(p + n)}.
\end{equation}

(4.4.10)

Using (4.4.9) and (4.4.10), we get

\begin{equation}
\left| \frac{f(z)}{k(z)} - 1 \right| \leq \frac{\sum_{k=p+n}^{\infty} |a_k - b_k|}{1 - \sum_{k=p+n}^{\infty} b_k} \\
\leq \frac{\delta}{(p+n)} \left[ 1 - \frac{\beta |b|[1 + \lambda(p-1)]}{[1 + \lambda(p + n - 1)][1 + \beta|b|]\Phi(p + n)} \right] \\
\leq \frac{\delta}{(p+n)} \left[ 1 + \lambda(p + n - 1)[1 + \beta|b|]\Phi(p + n) - \beta |b| \left[ 1 + \lambda(p-1) \right] \right] \tag{4.4.11}
\end{equation}

provided that \( \alpha \) is given by (4.4.8), thus by condition (4.4.3), \( f \in S^\alpha(b, \lambda, \beta; p) \), where \( \alpha \) is given by (4.4.8).

\begin{itemize}
  \item [\textbf{References}]
\end{itemize}
Subclasses of multivalent functions


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