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Subsemigroups of Cancellative Amenable Semigroups

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Abstract

We generalize a theorem of Frey by giving sufficient conditions for a subsemigroup T of a cancellative left amenable semigroup S to be left amenable. In particular, we show that if S is left amenable and T does not contain a free subsemigroup on two generators, then T is left amenable as well.

1 Introduction

Let S be a semigroup, and let $B(S)$ denote the set of all bounded, real valued functions on S. For each $f \in B(S)$ and for each $a \in S$, define $f_a \in B(S)$ by $f_a(x) = f(ax)$ for each $x \in S$. We say that S is *left amenable* if there exists a function $\mu : B(S) \to \mathbb{R}$ such that for all $f, g \in B(S)$, for each $a \in S$, and for all $r \in \mathbb{R}$:

1. $sup_{x \in S} f(x) \geq \mu(f) \geq inf_{x \in S} f(x)$
2. $\mu(f_a) = \mu(f)$
3. $\mu(f + g) = \mu(f) + \mu(g)$
4. $\mu(rf) = r\mu(f)$.

The function μ is called a *left invariant mean* on the semigroup S. A group G is called *amenable* if it is left amenable. Examples of amenable groups and semigroups include all commutative semigroups (in particular all abelian groups), all finite groups, and all solvable groups. Examples of nonamenable groups and semigroups include any free group or semigroup of rank two or higher. For more on amenable groups and semigroups, see [2], [3].

In [4], Følner proves the following theorem which states that the amenability of a group G is inherited by all of the subgroups of G.

Theorem 1. *Let G be a group, and let H be a subgroup of G. If G is amenable, then H is amenable.*

In [5], Frey gives an example of a left amenable semigroup S such that S contains a subsemigroup T which is not left amenable. In particular, Frey defines T to be the free group G_2 on two generators, and then forms S by adjoining an element 0 to T such that $00 = 0$ and such that for each $g \in T$, $0g = g0 = 0$. The semigroup S is left amenable since we can define a left invariant mean μ on S by $\mu(f) = f(0)$ [2].

Note that the semigroup S in the example given by Frey is not cancellative. However, in [6], Hochster gives an example of an amenable group which contains (and in fact, is generated by) a free subsemigroup on two generators. Thus, in general a subsemigroup T of a left amenable semigroup S need not inherit left amenability.

The examples given by Frey and Hochster both contain a free subsemigroup on two generators. In [5], Frey proves the following theorem.

Theorem 2. *Let S be a cancellative semigroup such that S contains no free subsemigroup on two generators. If S is left amenable, then every subsemigroup of S is left amenable.*

In Section 3, we generalize Theorem 2 by giving sufficient conditions for a subsemigroup T of a cancellative left amenable semigroup S to be left amenable. In particular, we show that if S is left amenable and T does not contain a free subsemigroup on two generators, then T is left amenable as well. Note that in our generalization of Theorem 2, S itself might contain a free subsemigroup on two generators.

In section 4, we give an example of a cancellative semigroup S which is not left amenable, but all of whose subsemigroups which themselves do not contain a free subsemigroup on two generators are left amenable. We then use a semigroup constructed in [2] by Day to give an example of a noncancellative left amenable semigroup S which contains a subsemigroup T such that T does not contain a free subsemigroup on two generators and such that T is not left amenable. Thus, our generalization of Theorem 2 only holds when the semigroup S is cancellative.

2 Background and Basic Definitions

In this section we state some theorems and definitions that we will need in section 3.

We say that a semigroup V has *common right multiples* if for each pair of elements $a, b \in V$, there exist elements $x, y \in V$ such that $ax = by$. A theorem of Ore states that if S is a cancellative semigroup which has common right multiples, then S embeds as a subsemigroup into a group G such that the following three conditions are satisfied:

(i) For each $g \in G$, there exists $x, y \in S$ such that $g = xy^{-1}$.

(ii) If S has semigroup presentation $\langle A \mid R \rangle$, then G is isomorphic to the group defined by the presentation $\langle A \mid R \rangle$.

(iii) The group G is amenable if and only if the semigroup S is left amenable .

The group G is called the *group of right fractions* of S, and S is called the *positive semigroup* of G. The following theorem was originally proven for groups by Følner, and was then generalized to cancellative semigroups by Frey [4], [5], [8].

Theorem 3. *A cancellative semigroup S is left amenable if and only if for each $\epsilon \in (0, 1)$, and for each finite, nonempty subset $H \subseteq S$, there exists a finite, nonempty subset $E \subseteq S$ such that for each $h \in H$,* $\dfrac{|hE \cap E|}{|E|} > \epsilon$

We call the set E in Theorem 3 a *Følner Set* of the semigroup S. The following lemma is well known and follows from Theorem 3.

Lemma 1. *Let S be a cancellative semigroup. If S is left amenable, then S has common right multiples.*

To see why Lemma 1 is true, assume that S is a cancellative left amenable semigroup, and let $a, b \in S$. Since S is a cancellative left amenable semigroup, then there exists a Følner set $E \subseteq S$ such that

$$\frac{|aE \cap E|}{|E|} > \frac{3}{4} \quad \text{and} \quad \frac{|bE \cap E|}{|E|} > \frac{3}{4}.$$

Thus, $aE \cap bE \neq \varnothing$, which implies that there exist $e_1, e_2 \in E$ such that $ae_1 = be_2$.

The following lemma of Frey is proven in [5].

Lemma 2. *Let S be a cancellative semigroup. Then S contains no free subsemigroup on two generators if and only if every subsemigroup T of S has common right multiples.*

The following lemma is proven in [2].

Lemma 3. *Let S and Q be semigroups. If S is left amenable, and $f : S \to Q$ is a semigroup homomorphism onto Q, then Q is left amenable.*

3 A Proof of the Main Theorem

In this section, we generalize Theorem 2 for cancellative semigroups by showing that if S is a cancellative left amenable semigroup and T is a subsemigroup of S such that T does not contain a free subsemigroup on two generators, then T is left amenable. Note that in Theorem 4 below, the semigroup S might contain a free subsemigroup on two generators. Note also that Theorem 2 follows as a corollary of Theorem 4. In particular, if a cancellative left amenable semigroup S contains no free subsemigroup on two generators, then given any subsemigroup T of S, T also contains no free subsemigroup on two generators, and therefore, by Theorem 4, we have that T is left amenable. We begin with the following lemma.

Lemma 4. *Let S be a cancellative semigroup which has common right multiples. Let T be a subsemigroup of S such that T also has common right multiples. Let H denote the group of right fractions of S, and let D denote the group of right fractions of T. Then T embeds into H, and D is isomorphic to a subgroup of H.*

Proof. Let T be given by the presentation

$$Q = \langle A \mid R \rangle$$

where A is a set of generators and R is a set of relations for T. Since D is the group of right fractions of T, then D also has presentation Q. In particular, A is also a generating set for D, and R is a set of defining relations for D. Since T is a subsemigroup of S and S embeds into H, then T also embeds into the group H. Since T embeds into H, and $A \subseteq T$, then A embeds into H. Given $u \in S$, then denote by $\hat{u}$ the copy of u which is embedded in H. Similarly, given $W \subseteq S$, then denote by $\hat{W}$ the copy of W which is embedded in H. Let K denote the subgroup of H generated by $\hat{A}$.

Since T embeds into H, then all relations of T, written using the elements of $\hat{A}$, hold in H. Since both T and D are defined by the presentation Q, then all the defining relations for D in R, again written using the elements of $\hat{A}$, hold in H. Since the relations in R are all in terms of the generators in $\hat{A}$, then these relations hold in K. Therefore, we can define a homomorphism $\phi : D \to K$ from D onto K by first defining a bijection $f : A \to \hat{A}$ from the generating set A of D to the embedded copy $\hat{A}$ of A in H, and then extending this bijection f to all of D by linearity. Thus, K is the homomorphic image under ϕ of the group D. Since the bijection f sends a generator $a \in A$ to its embedded copy $\hat{a} \in H$, and since T is embedded as a subsemigroup $\hat{T}$ in H, then for each element $z \in T$, ϕ sends z to its embedded copy $\hat{z}$ in H. Thus, for each $z \in T$, $\phi(z) = \hat{z}$ and $\phi(z)^{-1} = \hat{z}^{-1}$. Moreover, since $\hat{T}$ is an embedding of

T into K (by means of the homomorphism ϕ), then for each $c, d \in T$, we have that $c = d$ if and only if $\hat{c} = \hat{d}$.

We next show that ϕ is injective. Let $g, h \in D$. Assume that $\phi(g) = \phi(h)$. Let $x, y \in T$ be such that $xy^{-1} = gh^{-1}$. Since $\phi(g) = \phi(h)$, then we see that $\hat{x}\hat{y}^{-1} = \phi(x)\phi(y)^{-1} = \phi(xy^{-1}) = \phi(gh^{-1}) = \phi(g)\phi(h)^{-1} = 1_K$, where 1_K is the identity element of K. Therefore, $\hat{x}\hat{y}^{-1} = 1_K$, which implies that $\hat{x} = \hat{y}$. But this implies that $x = y$, which in turn implies that $gh^{-1} = xy^{-1} = 1_D$, where 1_D is the identity element of D. Thus, $g = h$. Hence, it follows that ϕ is an isomorphism from D to K. □

Theorem 4. *Let S be a cancellative semigroup. Let T be a subsemigroup of S such that T does not contain a free subsemigroup on two generators. If S is left amenable, then T is left amenable.*

Proof. Assume that S is left amenable. Since S is a cancellative, left amenable semigroup, then it follows by Lemma 1 that S has common right multiples. Thus, it follows by the theorem of Ore that S embeds as a subsemigroup into a group of right fractions H. Since S is cancellative and T is a subsemigroup of S, then it follows that T is cancellative as well. Since T is a cancellative semigroup which contains no free subsemigroup on two generators, then it follows by Lemma 2 that T has common right multiples. Thus, it follows by the theorem of Ore that T embeds as a subsemigroup into a group of right fractions D, and moreover, that the group D is amenable if and only if the semigroup T is left amenable. By Lemma 4, we have that the group D is isomorphic to a subgroup K of H. Since H is the group of right fractions of S, and since S is left amenable, then it follows by the theorem of Ore that H is amenable. Since H is amenable, and since K is a subgroup of H, then it follows by Theorem 1 that K is amenable. Since K is isomorphic to D, then it follows by Lemma 3 that D is amenable. Hence, T is left amenable. □

4 A Counter-Example to the Converse of Theorem 4

In the following example, we use the free semigroup $\mathcal{F}$ on two generators to show that the converse of Theorem 4 is false. In particular, we show that given any subsemigroup U of $\mathcal{F}$ such that U does not contain a free subsemigroup on two generators, then U is left amenable.

Example 1. Let $\mathcal{F}$ denote the free semigroup on two generators. Then $\mathcal{F}$ is not left amenable [2], [3], [5]. Let U be a subsemigroup of $\mathcal{F}$ such that U contains no free subsemigroup on two generators. Let $a, b \in U$. Let D denote the subsemigroup of U generated by a and b. Since $a, b \in U \subseteq \mathcal{F}$, then D

is also a subsemigroup of $\mathcal{F}$. Since U contains no free subsemigroup on two generators, then it follows that D is not a free subsemigroup of $\mathcal{F}$. Since D is a 2-generated subsemigroup of $\mathcal{F}$ which is not a free subsemigroup, then it follows that D is commutative [1], [7]. Thus, $ab = ba$. Since a and b were arbitrary elements of U, then it follows that U is commutative. Since U is commutative, then it follows that U is left amenable [2], [3].

The semigroup T used in the next example was constructed in [2] by Day. Here, we use the semigroup T to show that if a noncancellative left amenable semigroup S contains a subsemigroup T such that T does not contain a free subsemigroup on two generators, then it might not necessarily be the case that T is left amenable.

Example 2. Let $T = \{a_j \mid j \in \mathbb{Z} \text{ with } j \geq 1\}$. Define a multiplication on T by $a_p a_q = a_p$. That is, the product of any two elements is always equal to the element on the left. Note that the product $a_{i_1} a_{i_2} a_{i_3} \cdots a_{i_n}$ of any finite number of elements $a_{i_1}, a_{i_2}, a_{i_3}, \ldots a_{i_n} \in T$ is always equal to the leftmost element a_{i_1} in the product. We use this fact to see that T does not contain a free subsemigroup on two generators. Let $a_{i_1}, a_{i_2} \in T$, and let w_1 and w_2 be any two words over $\{a_{i_1}, a_{i_2}\}$. Then we see that $a_{i_1} w_1 = a_{i_1} = a_{i_1} w_2$. Thus, the semigroup generated by $\{a_{i_1}, a_{i_2}\}$ is not free. In [2], Day shows that the semigroup T is not left amenable.

Define the semigroup S, which contains T as a subsemigroup, in the following way. Let $S = \{e\} \cup T$, where e is not equal to any of the elements of T. We define the multiplication on S by letting the product of any two elements in T be as defined above, and for each $b \in S$, we define $eb = be = e$. Since for any distinct $a_i, a_j \in T$, we have that $ea_i = ea_j = e$, then it follows immediately that S is not cancellative. Following the method used by Day in [2], we see that the function $\mu : B(S) \to \mathbb{R}$ defined by $\mu(f) = f(e)$ is a left invariant mean on S. Thus, S is a noncancellative, left amenable semigroup which contains a subsemigroup T such that T contains no free subsemigroup on two generators and is not left amenable.

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