Multiplication Modules

F. Bakhshizade

Department of Mathematics, Natanz Branch
Islamic Azad University Natanz, Iran
Bakhshizadefateme2@gmail.com

Abstract

In this article we study the properties of multiplicative modules on commutative ring with identity. We will prove that a $R$-module $M$ is multiplicative if and only if for maximal ideal $P$ of $R$, $R$-module $M$ is $P$-cyclic or $P$-torsion.

Keywords: multiplication modules, P-cyclic and P-torsion modules

1 Introduction

We assume that all rings are commutative with identity and all modules are unitary. $R$-modules $M$ is called multiplication if for every submodule $N$ of $M$, there is an ideal $I$ of $R$ such that $N = IM$.

In the next section we will offer the conditions that are equivalent to multiplication.

2 Multiplicative Modules

Theorem 2.1. every homomorphism image of a multiplicative module is multiplicative.

Proof. It is obtained and proved stringhtly, and omit.

Definition 2.2. Let $M$ be a multiplicative $R$-module. We define multiplication on submodules, if $N$ and $K$ be two submodules of $M$ then there are ideals $I$ and $J$ such that $N = IM$, $K = JN$. We define $NK = IJM$.

In Thearem 2.3,We show that NK is independent of ideals of N and K.

Theorem 2.3. Let $R$ be a ring and $M$ be a multiplicative $R$-module. If $N$ and $K$ are submodules of $M$ then $NK$ is independent of $N$, $K$. 


Proof. Let there are ideals $I_i$ and $J_i$ $(i = 1, 2)$ of $R$ that $N = I_1M = I_2M$ and $K = J_1M = J_2M$, thus for every $r \in I_1$, $s \in J_1$ and $m \in M$, Since $sm \in J_1M = J_2M$, we have $sm = \sum_{i=1}^{n} r_i m_i$ where $n$ is a natural number and $r_i \in J_2$, $m_i \in M$, $(i = 1, \ldots, n)$ so $rsm = \sum_i +i = 1^n r_i(rm_i)$.

Since $rm_i \in I_1M = I_2M$ we conclude that there are $t_{ij} \in I_2$ and $m'_{ij} \in M (j = 1, \ldots, k_i)$ such that $r m_i = \sum_i +i = 1^k_i t_{ij} m'_{ij}$. Therefore $rsm = \sum_{i=1}^{n} \sum_{j=1}^{k_j} r_i t_{ij} m'_{ij}$, thus $rsm \in I_2J_2M$. so $I_1J_1M \subseteq I_2J_2M$.

Definition 2.4. Let $R$ be a ring and $P$ be a maximal ideal of $R$. If $M$ is an $R$-module, then we define:

1. $M$ is called $P$-torsion, If for every $m \in M$ there is $p \in P$ that $(1-p)m = 0$

2. $M$ is called $P$-cyclic if there are $x \in M$ and $q \in P$ such that $(1-q)M \subseteq Rx$.

Note that if $P$ is a maximal ideal of $R$ and $M$ is a $P$-torsion then $M_p = 0$

Before proving main theorem we need to prove some lemmas and offer some definitions. 

Definition 2.4. Let $N$ be a submodule of an $R$-module $M$. If $G$ is nonempty subset $M$, then $\{ r \in R \ \forall j \in G, r_j \in N \}$ is an ideal of $R$ that denoted by $(N : G)$. Note that if $K$ is a generated submodule by $G$ than $(N : G) = (N : K)$.

Definition 2.5. Let $R$ and $S$ be two Commutative rings with identity and $f : R \rightarrow S$ be a ring homomorphism.

for every ideal $I$ of $R$, ideal $f(I)S$ denoted by $I^e$ is called extension $I$ with $f$.

Lemma 2.6. Let $R$ be ring , $M$ be an $R$-module and $L, N$ be submodules of $M$ and be a closed multiplicative subset of $R$. If $N$ is finite generated then for natural homomorphism $R \rightarrow S^{-1}R$, we have $(L : _R N)^e = (S^{-1}L : S^{-1} _R S^{-1}N)$.

Lemma 2.7. Let $R$ be a ring and $M$ be an $R$-module and $L_1$, $L_2$ are submodules of $M$ as well as $S$ be closed multiplicative subset of $R$. If $I$ is an ideal of $R$ and $R \rightarrow S^{-1}R$ is natural homomorphism then the following statements are hold:

$$S^{-1}(IM) = I^e S^{-1}M2)S^{-1}(L_1 \cap L_2) = S^{-1}L_1 \cap S^{-1}L_2$$

Lemma 2.8. Let $R$ be a ring, $M$ be an $R$-module and $N$, $K$ be submodules of $M$. If $N_P = K_P$ for every maximal ideal $P$ of $R$, then $N = K$.

Lemma 2.9 (Nakayama). Let $R$ be a Commutative ring with identity, $M$ be an $R$-module and $I$ be an ideal of $R$ that $I \subseteq \text{Jac}(R)$. If $M = IM$ then $M = 0$. 

For every $R$-module $M$, suppose that $\Theta(M) = \sum_{m \in M} (Rm : M)$. If $M$ be a multiplicative module then

$$M = \sum_{m \in M} Rm = \sum_{m \in M} Rm = \sum_{m \in M} (Rm : M)M = \Theta(M)M.$$ 

**Theorem 2.10.** Let $R$ be a ring and $M$ be an $R$-module, The following statements are equivalent:

1. $M$ is multiplicative;
2. For every maximal ideal $P$ of $R$, If $\Theta(M) \subseteq P$ then $M_P = 0$;
3. For every submodule $N$ of $M$, $N = \Theta(M)N$;
4. For every maximal ideal $P$ of $R$ either $M$ is $P$-torsion or $M$ is $P$-cyclic.

**Proof.** (1 $\Rightarrow$ 2) we know $M = \Theta(M)M$. Let $P$ be a maximal ideal of $R$. If $\Theta(M) \subseteq P$ then for every $m \in M$,

$$Rm = (Rm : M)M = (Rm : M)\Theta(M)M = \Theta(M)(Rm : M)M = \Theta(M)Rm.$$ 

Thus according to Lemma 2.8 (1), We have $(Rm)_P = \Theta(M)_P(Rm)_P \Theta(Rm)_P$. therefore $(Rm)_P = P_P(Rm)_P$ Since $R_P$ is a local ring an $P_P$ its maximal ideal, according to lemma 2.10, $(Rm)_P = 0$, thus $M_P = 0$.

(2 $\Rightarrow$ 3 ) suppose that $N$ be a submodule of $M$. for every maximal ideal $P$ of $R$ there are two cases:

I) If $\Theta(M) \subseteq P$ then $(\Theta(M)N)_P \subseteq N_P \subseteq M_P = 0$, So $(\Theta(M)N)_P = N_P = 0$.

II) If $\Theta(M) \not\subseteq P$, then $\Theta(M)_P = R_P$, So according to Lemma 2.8 (1), we have $(\Theta(M)_PN)_P = \Theta(M)_PN_P = N_P$. thus with using Lemma 2.9, $\Theta(M)N = N$.

(3$\Rightarrow$4) Let $P$ be a maximal ideal of $R$. there are two cases:

I) $\Theta(M) \subseteq P$, since for every $m \in M$, $Rm$ is a submodule of $M$, according to assumption $\Theta(M)Rm = Rm$ so there is $p \in \Theta(M) \subseteq P$, that $m = pm$, Thus $(1-p)m = 0$, Therefore $M$ is $P$-torsion.

II) $\Theta(M) \not\subseteq P$. In this case there is $m \in M$ that $(Rm : M) \not\subseteq P$. thus $(Rm : M) + P = R$. hence there are $r \in (Rm : M)$ and $q \in P$ such that $r + q = 1$, so $1 - q = r \in (Rm : M)$ and $(1 - q)M \subseteq Rm$, therefore $M$ is $p$-cyclic. (4$\Rightarrow$1) It is clear.
References


Received: December, 2011