A Study on Fuzzy Complex Linear Programming Problems

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Abstract. In this paper we consider a certain kind of linear programming problems. This kind, in which fuzzy complex number is the coefficient of the objective function, is called fuzzy complex linear programming problem. Solutions of this kind of problems will be characterized by deriving its fuzzy Kuhn-Tucker conditions. The results include a duality relation between $\alpha$-cut programming and its dual are discussed. Also, we introduce the proof of Kuhn-Tucker stationary-point necessary optimality theorem for our problem.

Keywords: Cone, polyhedral cone, fuzzy set, fuzzy number, $\alpha$-cut programming fuzzy complex number, complex programming.

1. Introduction

Since the concept of fuzzy complex numbers was first introduced by J. J. Buckley in 1989 [6], many papers were devoted to studying the problems of the concept of fuzzy complex numbers. This new branch subject will be widely applied in fuzzy system theory, especially in fuzzy mathematical programming, and will also be widely applied in complex mathematical programming. It is well known that, complex programming, the extension to complex variables and functions of mathematical programming was initiated by N. Levinson in [9], where a duality theory for complex linear programming is given and the basic theorem of linear inequalities are extended to the complex case. Similar results were previously developed by Bellman and Fan in [1], for Hermitian matrices. This work of Levinson was continued by Hanson and Mond in [3], where it is

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considered as an extensions to quadratic and non linear programming. The fuzzy set theory has been applied to many disciplines such as control theory and management sciences, mathematical modeling, operations research and many industrial applications. The concept of fuzzy mathematical programming on general level was first proposed by Tanaka et al. [4]. In the framework of the fuzzy decision of Bellman and Zadeh [12]. Zimmermann [5] first introduced fuzzy linear programming as conventional linear programming. He considered problem with a fuzzy goal and fuzzy constraints, used linear membership functions and the min operator as an aggregator for these functions, and assigned an equivalent problem to fuzzy linear programming. This study focuses on fuzzy complex linear programming (FCLP) problems. Hence, first some important concepts on fuzzy and complex mathematical programming are mentioned.

This paper organized in 5 sections. In section 2, we give some necessary notations and definitions of fuzzy set theory, and complex mathematical programming problems. The concept of fuzzy complex numbers is introduced in section (3). Formulation of fuzzy complex linear programming, and deriving Kuhn-Tucker stationary point problem are presented in section 4. Finally, section 5 for conclusion.

2. Notations and Definitions

2.1. Notations:
- $\mathbb{C}^n (\mathbb{R}^n)$ the $n$-dimensional complex (real) vector space over the field $\mathbb{R}^n$.
- $\mathbb{C}^{m \times n} [\mathbb{R}^{m \times n}]$ the $m \times n$ complex [real] matrices.
- For $A = (a_{ij}) \in \mathbb{C}^{m \times n}$:
  - $\overline{A} = (\overline{a}_{ij})$ – the conjugate,
  - $A^T = (a_{ji})$ – the transpose,
  - $A^H = \overline{A}^T$ – the conjugate transpose.
- For $x = (x_i) \in \mathbb{C}^n, \ y \in \mathbb{C}^n$:
  - $\langle x, y \rangle = y^H x$ – the inner product of $x$ and $y$,
  - $|x| = (|x_i|) \in \mathbb{R}^n$ – the absolute value,
  - $\overline{x} = (\overline{x}_i)$ – the conjugate,
  - $\text{Re}(x) = (\text{Re}x_i) \in \mathbb{R}^n$ – the real part, of $x$
  - $\text{Im}(x) = (\text{Im}x_i) \in \mathbb{R}^n$ – the imaginary part, of $x$
  - $\text{arg}(x) = (\text{arg}x_i)$ – the argument of $x$.
- For a nonempty set $S \subset \mathbb{C}^n$:
  - $S^* = \{ y \in \mathbb{C}^n : x \in S \Rightarrow \text{Re}(y, x) \geq 0 \}$ – the dual (also polar) of $S$.
  - $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_i \geq 0 \ (i = 1, \ldots, n) \}$ the nonnegative orthant of $\mathbb{R}^n$
  - $x \succeq y$ denoted $x - y \in \mathbb{R}^n_+$, for $(x, y) \in \mathbb{R}^n$. 

Fuzzy complex linear programming problems

- For an analytic function $f : \mathbb{C}^n \rightarrow \mathbb{C}$ and a point $z^0 \in \mathbb{C}^n$
  \[ \nabla_z f (z^0) = \left( \frac{\partial f}{\partial z_i} (z^0) \right), \quad (i = 1, \ldots, n) \]
  the gradient of $f$ at $z^0$.

2.2. Definitions

Definition 2.2.1. [1]. A non-empty set $S \subset \mathbb{C}^n$ is:
  (a) convex if $0 \leq \lambda \leq 1 \Rightarrow \lambda S + (1 - \lambda) S \subset S$,  
  (b) a cone if $0 \leq \lambda \Rightarrow \lambda S \subset S$,
  (c) a polyhedral cone if for some positive integer $k$ and $A \in \mathbb{C}^{n \times k} :$
  \[ S = A R^k_+ = \{ Ax : x \in R^k_+ \} \], i.e. $S$ is generated by finitely many vectors (the columns of $A$).

Definition 2.2.2. [7]. A fuzzy set $\tilde{a}$ in $\mathbb{R}$ is a set of ordered pairs:
  \[ \tilde{a} = \{ (x, \mu_\tilde{a}(x)) : x \in \mathbb{R} \} \],
  $\mu_\tilde{a}(x)$ is called the membership function of $x$ in $\tilde{a}$
  
  if $\mu_\tilde{a}(x) = 1$, the fuzzy set $\tilde{a}$ is called normal.

Definition 2.2.3. [7]. The support of a fuzzy set $\tilde{a}$ on $\mathbb{R}$ is the crisp set of all $x \in \mathbb{R}$ such that $\mu_\tilde{a}(x) > 0$.

Definition 2.2.4. The $\alpha$-level set (cut set) of a fuzzy set $\tilde{a}$ is defined as an ordinary set $a_\alpha$ for which the degree of its membership function exceeds the level $\alpha$:
  \[ a_\alpha = \{ x : \mu_\tilde{a}(x) \geq \alpha \}, \quad \alpha \in [0, 1] \].

Definition 2.2.5. [4]. A fuzzy set $\tilde{a}$ on $\mathbb{R}$ is convex if
  \[ \mu_\tilde{a}(\lambda x + (1 - \lambda) y) \geq \min \{ \mu_\tilde{a}(x), \mu_\tilde{a}(y) \}, \quad x, y \in \mathbb{R}, \quad \lambda \in [0, 1] \].
  Note that, a fuzzy set is convex if all $\alpha$-cuts are convex.

Definition 2.2.6. [5]. A fuzzy number $\tilde{a}$ is a convex normalized fuzzy set on the real line $\mathbb{R}$ such that:
  (1) It exists at least one $x_0 \in \mathbb{R}$ which $\mu_\tilde{a}(x_0) = 1$,
  (2) $\mu_\tilde{a}(x)$ is piecewise continuous.

From the definition of a fuzzy number $\tilde{a}$, it is significant to note that, the $\alpha$-level set $a_\alpha$ of a fuzzy number $\tilde{a}$ can be represented by the closed interval which depends on the value of $\alpha$. Namely,
  \[ a_\alpha = \{ x \in \mathbb{R} : \mu_\tilde{a}(x) \geq \alpha \} = [a^-(\alpha), a^+ (\alpha)] \],
  where $a^-(\alpha)$ or $a^+(\alpha)$ represents the left or right extreme point of the $\alpha$-level set $a_\alpha$, respectively.

3. Fuzzy Complex Numbers [2, 6, 8, 13]

The concept of fuzzy complex numbers was first introduced by J. J. Buckley in 1989, many papers were devoted to studying the problems of the concept of fuzzy
complex numbers. Uncertainty of complex valued physical quantities \( c = a + ib \) can be described by complex fuzzy sets, such sets can be described by membership functions \( \mu(a, b) \) which map the universe of discourse (complex plane) into the interval \([0, 1]\). The problem with this description is that it is difficult to directly translate into words from natural language. To make this translation easier, several authors have proposed to use, instead of a single membership function for describing the complex number, several membership function which describe different real valued characteristics of this number, such as its real part, its imaginary part, its absolute value, etc. Thus, a natural idea is to represent a complex fuzzy number by describing two real fuzzy numbers: \( \tilde{a} \) and \( \tilde{b} \) characterized by the corresponding membership functions \( \mu_{\tilde{a}}(a), \mu_{\tilde{b}}(b) \). In this approach, for every complex value \( a + ib \), i.e., for every pair \((a, b)\), the degree \( \mu_{\tilde{a}}(a, b) \) with which this complex value is possible can be defined as
\[
\mu_{\tilde{a}}(c) = \mu_{\tilde{a}}(a, b) = \min\left( \mu_{\tilde{a}}(a), \mu_{\tilde{b}}(b) \right).
\]

### 3.1 Preliminary Concepts

Let \( \mathbb{R} \) be a set of real numbers, \( C = \{x + iy : x \in \mathbb{R}, y \in \mathbb{R}, i = \sqrt{-1}\} \) a field of complex numbers, a finite closed interval \( X = [X^-, X^+] \) is called a closed interval number on \( \mathbb{R} \), \( I(\mathbb{R}) \) denotes the set of all closed interval numbers on \( \mathbb{R} \).

For arbitrary intervals \( X = [X^-, X^+] \), \( Y = [Y^-, Y^+] \in \mathbb{R} \), \( Z = X + iY = \{x + iy \in C : x \in X, y \in Y, i = \sqrt{-1}\} \) is called a closed complex interval number, \( I(C) = \{Z = X + iY : X, Y \in I(\mathbb{R}), i = \sqrt{-1}\} \) denotes the set of all closed complex interval numbers on \( C \), *' \( = \{+, -, \times, \div\}' \) is a binary operation on \( I(C) \), (when *' \( = \{+, -\}' \) is a binary operation on \( I(\mathbb{R}) \), \( \forall k \in \mathbb{R}^+ = [0, +\infty) \), let \( k \cdot X = [k X^-, k X^+] \), let \( k \cdot Z = (k X^+ + i(k Y^+) = [k X^-, k X^+] + i[k Y^- + k Y^+] \), \( k \cdot Z = X + iY \in I(C) \), \( Z \subseteq C \) and \( Z = \{x + iy \in C : x \in X, y \in Y\} \), then \( Z \) is a convex set on \( C \).

**Definition 3.2.** Let \( C \) be a field of complex number, mapping \( \tilde{Z} : C \rightarrow [0, 1] \) is called a fuzzy complex set, \( \tilde{Z}(z) \) is called the membership function of fuzzy set \( \tilde{Z} \) for \( z \), \( F(C) = \{\tilde{Z} : \tilde{Z} : C \rightarrow [0, 1]\} \) denotes all fuzzy complex sets on \( C \).

**Definition 3.3.** \( \tilde{Z}_\alpha = \{z = x + iy \in C : \tilde{Z}(z) = \tilde{Z}(x + iy) \geq \alpha \} \) is called \( \alpha \)-cut set of \( \tilde{Z} \).
Definition 3.4. \( \tilde{Z}_0 = \text{supp} \tilde{Z} = \{ z = x + iy \in C : \tilde{Z}(z) = \tilde{Z}(x + iy) > 0 \} \) is called support set of \( \tilde{Z} \).

Definition 3.5. \( \tilde{Z} \in F(C) \) is called a convex fuzzy complex set on \( C \), if and only if for all \( \alpha \in [0,1] \), \( \tilde{Z}_\alpha \) is a convex complex set on \( C \).

Definition 3.6. \( \tilde{Z} \in F(C) \) is a normal fuzzy complex set on \( C \), if and only if \( \{ z \in C : \tilde{Z}(z) = \tilde{Z}(x + iy) = 1 \} \neq \emptyset \).

Definition 3.7. A normal convex fuzzy complex set on \( C \) is called a fuzzy complex number.

3.2. Buckley's Membership Function Description.

In order to describe a complex number \( C = a + ib \), we must describe two real numbers: its real part \( a \) and its imaginary part \( b \). Thus, a natural idea is to represent a complex fuzzy number \( \tilde{c} = \tilde{a} + i \tilde{b} \), by describing two real fuzzy numbers: \( \tilde{a} \) and \( \tilde{b} \) (see [2]) characterized by the corresponding membership functions \( \mu_\alpha(a) \) and \( \mu_\alpha(b) \). In this approach, for every complex value \( a + ib \), i.e., for every pair \( (a,b) \), the degree \( \mu_\alpha(c) = \mu_\alpha(a,b) \) with which this complex value is possible can be defined as \( \mu_\alpha(c) = \min(\mu_\alpha(a), \mu_\alpha(b)) \).

Then, for each \( \alpha \in [0,1] \), the \( \alpha \)-cut for the real part \( \tilde{a} \) is an interval \( [\alpha^-(\alpha), \alpha^+(\alpha)] \), the \( \alpha \)-cut for the imaginary part is also an interval \( [b^-(\alpha), b^+(\alpha)] \), and hence, the \( \alpha \)-cut for the resulting 2-D membership function \( \mu_\alpha(c) = \min(\mu_\alpha(a), \mu_\alpha(b)) \), is a rectangular "box" \( [\alpha^-(\alpha), \alpha^+(\alpha)] \times [b^-(\alpha), b^+(\alpha)] \). The boundary of this box consists of two straight line segments, which are parallel to the \( a \)-axis, and of two straight line segments which are parallel to the \( b \)-axis.

4. Fuzzy Complex Linear Programming

Consider the complex linear programming problem [1, 3, 9, 10]

\[
\begin{align*}
\min & \quad \text{Re}(c, x) \\
\text{s.t} & \quad Ax - b \in T, \quad x \in S,
\end{align*}
\]

where, \( A \in C^{m \times n} \), \( c \in C^n \), and let \( S \subset C^n \) and \( T \subset C^m \) be closed convex cones.
The fuzzy complex linear programming can be formulated as:

\[
\begin{align*}
\min & \quad \text{Re}(\hat{c}, x) \\
\text{s.t.} & \quad A x - b \in T, \quad x \in S,
\end{align*}
\]

(\(\tilde{P}\))

where \(\hat{c} = \hat{a} + i \hat{b}\) is a fuzzy complex number, \(\mu_{\hat{c}}(c) = \min(\mu_{\hat{a}}(a), \mu_{\hat{b}}(b))\), for each \(\alpha \in [0, 1]\), the \(\alpha\)-cut for the real part \(\hat{a}\) is an interval \([\alpha^{-}(\hat{a}), \alpha^{+}(\hat{a})]\), and the \(\alpha\)-cut for \(\hat{b}\) is \([\alpha^{-}(\hat{b}), \alpha^{+}(\hat{b})]\), and hence the \(\alpha\)-cut for the resulting 2-D membership function \(\mu_{\hat{c}}(c) = \mu_{\hat{c}}(a, b) = \min(\mu_{\hat{a}}(a), \mu_{\hat{b}}(b))\) is

\[
[\alpha^{-}(\hat{a}), \alpha^{+}(\hat{a})] \times [\alpha^{-}(\hat{b}), \alpha^{+}(\hat{b})],
\]

and it can be written in the following form

\[
L_{\alpha} = \{c : \mu_{\hat{c}}(c) \geq \alpha\},
\]

where \(c = a + ib\), \(\mu_{\hat{c}}(c) = \min(\mu_{\hat{a}}(a), \mu_{\hat{b}}(b))\), \(\alpha = (\alpha_1, \alpha_2)\), \(\mu_{\hat{a}}(a) \geq \alpha_1\) and \(\mu_{\hat{b}}(b) \geq \alpha_2\).

The fuzzy complex linear programming (\(\tilde{P}\)) can be converted to the deterministic \(\alpha\)-cut programming as

\[
\begin{align*}
\min & \quad \text{Re}(\hat{c}, x) \\
\text{s.t.} & \quad A x - b \in T, \quad x \in S, \\
& \quad c \in L_{\alpha}.
\end{align*}
\]

Choose \(\hat{c} \in L_{\alpha}\) and characterize the solution of (\(P_{\alpha}\)) corresponding to \(\hat{c}\), i.e. we will solve the problem

\[
\begin{align*}
\min & \quad \text{Re}(\hat{c}, x) \\
\text{s.t.} & \quad A x - b \in T, \quad x \in S,
\end{align*}
\]

(\(P\))

The dual problem of problem (\(P\)) is

\[
\begin{align*}
\max & \quad \text{Re}(b, y) \\
\text{s.t.} & \quad \hat{c} - A^H y \in S^*, \quad y \in T^*
\end{align*}
\]

(\(D\))

where \(S^*\) and \(T^*\) are dual cones of \(S\) and \(T\) respectively.

**Definition 4.1.** A vector \(x^0 \in \mathbb{C}^n\) is

(i) a feasible solution of (\(P\)) is

\[A x^0 - b \in T, \quad x^0 \in S,\]

(ii) an optimal solution of (\(P\)) if \(x^0\) is feasible and \(\text{Re}(\hat{c}, x^0) = \min\{\text{Re}(\hat{c}, x) ; x \text{ is feasible}\}\).

**Definition 4.2.** The problem (\(P\)) is

(i) consistent if it has feasible solutions,

(ii) unbounded if it is consistent, and if it has feasible solutions \(\{x_k ; k = 1, 2, ...\}\) with \(\text{Re}(\hat{c}, x_k) \to -\infty\).
Consistency and boundedness of \((D)\) and feasibility and optimality of its solutions, are similarly defined.

**Definition 4.3.** The Lagrangian of the problems \((P)\) and \((D)\) is

\[
L(x, y) = \Re\{(\tilde{c}, x) - (A x - b)\} = \Re\{(b, y) + (\tilde{c} - A^H y, x)\}
\]

**Definition 4.4.** The point \((x^0, y^0) \in S \times T^*\) is a saddle point of \(L(x, y)\) with respect to \(S \times T^*\) if

\[
L(x, y) \leq L(x^0, y^0) \leq L(x^0, y) \quad \text{for all} \quad x \in S, y \in T^*.
\]

A duality relations between \((P)\) and \((D)\) and a characterization of there optimal solutions, (if such solutions exist) are given in the following theorem.

**Theorem 4.1.** [6]. Let \(S\) and \(T\) in problems \((P)\) and \((D)\) be polyhedral cones, then:

(a) If one of the problems is inconsistent then, the other is inconsistent or unbounded.

(b) Let the two problems be consistent, and let \(x^0\) be a feasible solution of \((P)\) and \(y^0\) be a feasible of \((D)\). Then \(\Re(c, x^0) \geq \Re(b, y^0)\).

(c) If both \((P)\) and \((D)\) are consistent, then they have optimal solutions and their optimal values are equal.

(d) Let \(x^0\) and \(y^0\) be feasible solutions of \((P)\) and \((D)\) respectively. Then \(x^0\) and \(y^0\) are optimal if and only if

\[
\Re\{(A x^0 - b, y^0) + (\tilde{c} - A^H y^0, x^0)\} = 0,
\]

or equivalently if and only if

\[
\Re\{(A x^0 - b, y^0) = (\tilde{c} - A^H y^0, x^0) = 0.
\]

(e) The vectors \(x^0 \in C^n\) and \(y^0 \in C^m\) are optimal solutions of \((P)\) and \((D)\) respectively if and only if the point \((x^0, y^0)\) is a saddle point of \(L(x, y)\) with respect to \(S \times T^*\), in which case

\[
L(x^0, y^0) = \Re(\tilde{c}, x^0) = \Re(b, y^0).
\]

Now, let us return to the problem \((P_\alpha)\), suppose that \(x^0\) is a solution, the problem becomes:

\[
\begin{align*}
\min & \quad \Re(c, x^0) \\
\text{s.t} & \quad A x^0 - b \in T, x^0 \in S, c \in L_\alpha,
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\min & \quad \Re(c, x^0) \\
\text{s.t} & \quad c \in L_\alpha,
\end{align*}
\]

where \(L_\alpha = \{c : \mu_\alpha(c) \geq \alpha\}, \alpha = (\alpha_1, \alpha_2), \tilde{c} = \tilde{a} + i \tilde{b}, \mu_\alpha(c) = \min\{\mu_\alpha(a), \mu_\alpha(b)\}\).

The problem becomes
\[
\begin{align*}
\min \Re (c, x^0) \\
s.t \quad \alpha - \mu_c (c) \leq 0,
\end{align*}
\]
where \( \mu_c (c) = \min (\mu_a (a), \mu_b (b)) \), \( \alpha = (\alpha_1, \alpha_2) \), \( \mu_a (a) \geq \alpha_1, \mu_b (b) \geq \alpha_2 \).

We can define the following problems:

1- The minimization problem \((MP)\):
\[
\begin{align*}
\text{MP} \quad \begin{cases}
\text{Find } c^* \in \mathcal{L}_\alpha, \text{ if it exists, such that } \\
\Re (c^*, x^0) = \min_{c \in \mathcal{L}_\alpha} \Re (c, x^0), \quad c^* \in \mathcal{L}_\alpha = \{ c : \alpha - \mu_c (c) \leq 0 \}.
\end{cases}
\end{align*}
\]

2- The local minimization problem \((LMP)\):
\[
\begin{align*}
\text{LMP} \quad & \begin{cases}
\text{Find an } c^* \text{ in } \mathcal{L}_\alpha, \text{ if it exists, such that for some open ball } B_{\delta} (c^*) \\
\quad \text{around } c^* \text{ with radius } \delta > 0 \\
\quad c \in B_{\delta} (c^*) \cap \mathcal{L}_\alpha \Rightarrow \Re (c^*, x^0) \leq \Re (c, x^0)
\end{cases}
\end{align*}
\]

3- The Kuhn-Tucker stationary-point problem \((KTP)\): Find \( c^* \in \mathcal{L}_\alpha, u^* \in \mathbb{R}^{\mathcal{n}} \), if they exist, such that
\[
\begin{align*}
\psi (c^*, u) = & \Re (c^*, x^0) + u \alpha - \mu_c (c^*), \\
\quad \nabla_c \psi (c^*, u^*) = 0 \Rightarrow & \Re \nabla_c (c^*, x^0) - u^* \nabla_c \mu_c (c^*) = 0, \\
\quad \nabla_u \psi (c^*, u^*) \leq 0 \Rightarrow & \alpha - \mu_c (c^*) \leq 0, \\
\quad u^* \nabla_u \psi (c^*, u^*) = 0 \Rightarrow & u^* \left[ \alpha - \mu_c (c^*) \right] = 0, \\
\quad u^* \geq 0.
\end{align*}
\]

In the following, we introduce some constraint qualifications, which, we need it in our study, see [8]

1- Slater's Constraint Qualification.
Let \( \mathcal{L}_\alpha \) be a convex set in \( \mathbb{C}^n \), and \( \mu_c (c) \) be concave on \( \mathcal{L}_\alpha \).
\( \alpha - \mu_c (c) \leq 0 \) is said to satisfy slater's constraint qualification if there exists an \( c^* \in \mathcal{L}_\alpha \) such that \( \alpha - \mu_c (c^*) < 0 \).

2- Karlin's Constraint Qualification.
Let \( \mathcal{L}_\alpha \) be a convex set in \( \mathbb{C}^n \), and \( \mu_c (c) \) is concave on \( \mathcal{L}_\alpha \).
\( \alpha - \mu_c (c) \leq 0 \) is said to satisfy Karlin's constraint qualification if there exists no \( P \in \mathbb{R}, P \geq 0 \) such that \( P \alpha - P \mu_c (c) \geq 0 \) for all \( c \in \mathcal{L}_\alpha \).

3- The Strict Constraint Qualification.
Let \( \mathcal{L}_\alpha \) be a convex set in \( \mathbb{C}^n \), and \( \mu_c (c) \) is concave on \( \mathcal{L}_\alpha \).
\( \alpha - \mu_c (c) \leq 0 \) is said to satisfy strict constraint qualification if \( \alpha - \mu_c (c) \) is strictly convex at distinct points \( c^1, c^2 \in \mathcal{L}_\alpha \).

4- The Kuhn-Tucker Constraint Qualification.
Let \( \mathcal{L}_\alpha \) be an open set in \( \mathbb{C}^n \), and \( \mu_c (c) \) defined on \( \mathcal{L}_\alpha \), \( \alpha - \mu_c (c) \leq 0 \) is said to satisfy the Kuhn-Tucker constraint qualification at \( c^* \in \mathcal{L}_\alpha \) if \( \alpha - \mu_c (c) \) is
differentiable at $c^*$ and if $y \in \mathbb{C}^n$, $\nabla_c \mu_c(e)(c) \cdot \Re y \leq 0$ implies there exists an $n$-dimensional vector function $e$ defined on the interval $[0, 1]$ such that

(a) $e(0) = c^*$,

(b) $e(\tau) \in L_\alpha$ for $0 \leq \tau \leq 1$,

(c) $e$ is differentiable at $\tau = 0$ and $\frac{d}{d \tau}(e(0)) = \lambda y$ for some $\lambda > 0$.

5- The Arrow-Hurwicz-Uzawa Constraint Qualification.

Let $L_\alpha$ be an open set in $\mathbb{C}^n$, and $\mu_c(c)$ defined on $L_\alpha$, $\alpha - \mu_c(c)$ is said to satisfy the Arrow-Hurwicz-Uzawa constraint qualification at $c^* \in L_\alpha$ if $\alpha - \mu_c(c)$ is differentiable at $c^*$ and if $\nabla_c \mu(c) \Re z$ has a solution $z \in \mathbb{C}^n$.

6- The Reverse Convex Constraint Qualification.

Let $L_\alpha$ be an open set in $\mathbb{C}^n$, and $\alpha - \mu_c(c)$ defined on $L_\alpha$, $\alpha - \mu_c(c)$ is said to satisfy the Arrow-Hurwicz-Uzawa constraint qualification at $c^* \in L_\alpha$ if $\alpha - \mu_c(c)$ is differentiable at $c^*$, and either $\alpha - \mu_c(c)$ is concave at $c^*$ or $\alpha - \mu_c(c)$ is linear in $\mathbb{C}^n$.

**Theorem 4.2.** Let $L_\alpha$ be an open subset of $\mathbb{C}^n$, let $\Re(c, x^0)$ be convex and $\mu_c(c)$ be concave on $L_\alpha$, $c^* \in L_\alpha$ solve the problem

$$\min \Re(c, x^0)$$

s. t.

$$c \in L_\alpha = \{c : \alpha - \mu_c(c) \leq 0\},$$

and let $\Re(c, x^0)$ and $\alpha - \mu_c(c)$ be differentiable at $c^*$ and let $\alpha - \mu_c(c)$ satisfies one of the constraint qualification 1-6. Then, there exists a $u^* \in \mathbb{C}^n$ such that $(c^*, u^*)$ satisfies:

$$\Re \nabla_c(c^*, x^0) - u^* \nabla \mu_c(c^*) = 0,$$

and

$$u^* \{\alpha - \mu_c(c^*)\} = 0.$$

**Proof:** Let $c^*$ solve LMP with $\delta = \delta$. If $\alpha - \mu_c(c^*) < 0$, then, for $y \in \mathbb{C}^n$, $y^Ty = 1$, we have:

$$\alpha - \mu_c(c^* + \delta y) = \alpha - \mu_c(c^*) - \delta [\nabla \mu_c(c^*)y + \beta(c^*, \delta y)]$$

$$\Rightarrow \mu_c(c^* + \delta y) - \mu_c(c^*) = \delta [\nabla \mu_c(c^*)y + \beta(c^*, \delta y)].$$

Since $\alpha - \mu_c(c^*) < 0$ and $\lim_{\delta \to 0} \beta(c^*, \delta y) = 0$,

$$0 < \delta < \delta < \delta, \alpha - \mu_c(c^* + \delta y) < 0$$

and $c^* + \delta y \in L_\alpha$. But $c^*$ solves LMP, so

$$0 \leq \Re(c^* + \delta y, x^0) - \Re(c^*, x^0) = \delta [\Re \nabla_c(c^*, x^0)y + \beta(c^*, \delta y)]$$

for $0 < \delta < \delta$. Hence

$$\Re \nabla_c(c^*, x^0)y + \beta(c^*, \delta y) \geq 0.$$
Since $y$ is an arbitrary vector in $C^n$ satisfying $y^Ty=1$ we conclude from this last inequality, by taking $y=\pm e^i$, where $e^i \in C^n$ is a vector with one in the $i^{th}$ position and zero elsewhere, that
\[
\text{Re} \nabla_c (c^*, x^0) = 0.
\]
Hence $c^*$ and $u^* = 0$ satisfy that
\[
\text{Re} \nabla_c (c^*, x^0) - u^* \nabla_c (c^*, x^0) = 0, \quad \text{and}
\]
\[
u^* \left[ \alpha - \mu^* (c^*) \right] = 0.
\]
If $\alpha - \mu^* (c^*) = 0$. Let $\alpha - \mu^* (c)$ satisfy the Kuhn-Tucker constraint qualification at $c^*$, and let $y \in C^n$ satisfy
\[-\nabla_c (c^*) \text{Re} y \leq 0,
\]
there exists an $n$-dimensional vector function $e$ defined on $[0, 1]$ such that
\[
e(0) = c^*, \quad e(\tau) \in L_a, \quad \text{for } 0 \leq \tau \leq 1,
\]
e is differentiable at $\tau = 0$, and
\[
d \left( \frac{d (e(0))}{d \tau} \right) = \lambda \text{Re} y \quad \text{for some } \lambda > 0. \quad \text{Hence for } 0 \leq \tau \leq 1
\]
\[
e_i (\tau) = e_i (0) + \tau \left[ \frac{d e_i (0)}{d \tau} + \gamma_i (0, \tau) \right] \quad \text{for } i = 1, \ldots, n.
\]
where $\lim_{\tau \to 0} \gamma_i (0, \tau) = 0$. Hence by taking $\tau$ small enough, say $0 < \tau < \hat{\tau} < 1$, we have that $e(\tau) \in B_{\hat{\tau}} (c^*)$ since $e(\tau) \in L_a$, for $0 \leq \tau \leq 1$ and $c^*$ solves $LMP$, we have that
\[
\text{Re} (e(\tau), x^0) \geq \text{Re} (e(0), x^0) \quad \text{for } 0 < \tau < \hat{\tau}.
\]
Hence by the chain rule and the differentiability of $\text{Re} (e, x^0)$ at $c^*$, we have for $0 < \tau < \hat{\tau}$ that
\[
0 \leq \text{Re} (e(\tau), x^0) - \text{Re} (e(0), x^0) = \text{Re} \nabla (e(0), x^0) \frac{d e(0)}{d \tau} \tau + \beta(0, \tau) \tau,
\]
where $\lim_{\tau \to 0} \beta(0, \tau) = 0$. Hence
\[
\text{Re} \nabla (e(0), x^0) \frac{d e(0)}{d \tau} \tau + \beta(0, \tau) \tau \geq 0 \quad \text{for } 0 < \tau < \hat{\tau}.
\]
Taking the lim as $\tau$ approaches zero gives
\[
\text{Re} \nabla (e(0), x^0) \frac{d e(0)}{d \tau} \tau \geq 0.
\]
Since $e(0) = c^*$ and $\frac{d e(0)}{d \tau} = \lambda \text{Re} y$, for some $\lambda > 0$, we have that:
\[
\text{Re} \nabla_c (c^*, x^0 y) \geq 0.
\]
Hence, we have shown that if
\[
\text{Re} \left( -\nabla_c \mu_{\hat{\tau}} (c^*) y \right) \leq 0 \Rightarrow \text{Re} \nabla_c (c^*, x^0 y) \geq 0,
\]
or that
\[
\begin{cases}
\text{Re} \nabla_c (c^*, x^0 y) < 0 \\
\text{Re} \left( -\nabla_c \mu_{\hat{\tau}} (c^*) y \right) \leq 0
\end{cases}
\]
has no solution $y \in C^n$. 

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Hence by Motzkin's theorem of alternative in [7], there exists an $r_0^*$ and $r^*$ such that
\[ r_0^* \text{Re} \nabla_c (c^*, x^0) - r^* \nabla_c \mu_c (c^*) = 0 \quad \text{or} \quad r_0^* \text{Re} \nabla_c (c^*, x^0) - \frac{r^*}{r_0^*} \nabla_c \mu_c (c^*) = 0, \]

$r_0^* \geq 0$, $r^* \geq 0$. Since $r_0^*$ is a real number, $r_0^* \geq 0$ means $r_0^* > 0$, then by defining
\[ u^* = \frac{r^*}{r_0^*}, \]
we have that
\[ \text{Re} \nabla_c (c^*, x^0) - u^* \nabla_c \mu_c (c^*) = 0, \quad \text{and} \]
\[ u^* (\alpha - \mu_c (c^*)) = 0. \]

5. Conclusions

In this work, we can find a certain kind of mathematical programming problems called fuzzy complex linear programming by using the concept of fuzzy complex number in complex mathematical programming and $\alpha$-cut of a fuzzy complex number. Also, we introduce the proof of Kuhn-Tucker stationary-point necessary optimality theorem for our problem.

References


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