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Curious Congruences for Balancing Numbers

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Abstract

The balancing numbers are terms of a sequence defined in a quite simple recursive fashion. However, despite its simplicity, they have some curious properties, which are worth mentioning. In this paper, we establish some fascinating congruences involving balancing and related numbers.

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1 Introduction

Recently, Behera and Panda [1] introduced balancing numbers as solutions of the Diophantine equation $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$ calling $r \in Z^+$, the balancer corresponding to the balancing number n. The numbers 6,35,204 are examples of first three balancing numbers with balancers 2,14,and 84 respectively. In [3], Panda and Ray introduced cobalancing numbers by slightly modifying the original definition. They defined the cobalancing numbers as the solution of the Diophantine equation $1 + 2 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r)$ calling $r \in Z^+$, the cobalancer corresponding to the cobalancing number n. It is shown in [1] that a positive integer n is a balancing number if and only if n^2 is a triangular number or equivalently, $8n^2 + 1$ is a perfect square.Though the definition does not allow any balancing number to be less than 1, Panda [5, 6] and Panda and Ray [4] accepted 1 as a balancing number being the positive square root of the square triangular number 1. While the sequence $\{B_n\}_{n=1}^{\infty}$ of balancing numbers obey the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$. If n is a balancing number, C_n is called a Lucas-balancing number, while for a cobalancing number n, c_n is called a Lucas-cobalancing number [5]. The sequence of Lucas-balancing numbers satisfy the recurrence relation identical with that for balancing numbers, where as

the recurrence relation for Lucas-cobalancing numbers are not identical with that for cobalancing numbers and surprisingly, they are identical with that for balancing numbers [5]. Liptai [2] added another interesting result to the theory of balancing numbers by proving that the only balancing number in the Fibonacci sequence is 1. Also in [3] he proved that there are no Lucas balancing numbers in the sequence of Fibonacci numbers. In [7], Ray obtain nice product formulas for balancing and Lucas-balancing numbers.
It is proved in [1],that $\lim_{n\to\infty} \frac{B_{n+1}}{B_n} = 3+\sqrt{8}$. Throughout this paper, we denote $\lambda_1 = 3+\sqrt{8}$ and its reciprocal $\lambda_2 = 3-\sqrt{8}$.

2 Some Basic Properties of Balancing Numbers

If B_n and C_n be the n^{th} balancing and Lucas-balancing numbers, their closed forms, popularly known as Binet's formulas [1,4] are respectively given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{2\sqrt{8}} \tag{1}$$

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}. \tag{2}$$

Using (1) and (2), for all positive integer n, one can easily obtain the following important results.

$$B_{-n} = -B_n, C_{-n} = C_n \tag{3}$$

$$C_n = B_{n+1} - 3B_n \tag{4}$$

$$B_{n+1} - B_{n-1} = 2C_n \tag{5}$$

$$C_{n+1} = 3C_n + 8B_n \tag{6}$$

$$C_n = 3B_n - B_{n-1} \tag{7}$$

Keeping all these important results in mind, we suggest the proof of the following theorem.

Theorem 2.1 *If* $\binom{m}{j}$ *denote the usual notation for combination and for every integers* k, m, n, s *with* $m \geq 0$*, we have*

$$B_s^m B_{km+n} = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} B_k^j B_{k-s}^{m-j} B_{js+n} \tag{8}$$

and

$$B_s^m C_{km+n} = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} B_k^j B_{k-s}^{m-j} C_{js+n} \tag{9}$$

Proof. As $\lambda_1 + \lambda_2 = 6$, $\lambda_1\lambda_2 = 1$ and by (3), applying binomial theorem, we obtain
$\sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} B_k^j B_{k-s}^{m-j} B_{js+n}$
$= \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} (\frac{\lambda_1^k - \lambda_2^k}{2\sqrt{8}})^j (\frac{\lambda_1^{k-s} - \lambda_2^{k-s}}{2\sqrt{8}})^{m-j} (\frac{\lambda_1^{js+n} - \lambda_2^{js+n}}{2\sqrt{8}})$
$= \frac{1}{(\lambda_1-\lambda_2)^{m+1}} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} (\lambda_1^k - \lambda_2^k)^j (\lambda_1^{k-s} - \lambda_2^{k-s})^{m-j} (\lambda_1^{js+n} - \lambda_2^{js+n})$
$= \frac{1}{(\lambda_1-\lambda_2)^{m+1}} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} (\lambda_1^{js+n} - \lambda_2^{js+n}) (\lambda_1^k - \lambda_2^k)^j (\lambda_1^s\lambda_2^k - \lambda_1^k\lambda_2^s)^{m-j}$
$= \frac{1}{(\lambda_1-\lambda_2)^{m+1}} [\lambda_1^n \sum_{j=0}^{m} (\lambda_1^{s+k} - \lambda_1^s\lambda_2^k)^j (\lambda_1^s\lambda_2^k - \lambda_1^k\lambda_2^s)^{m-j}$
$-\lambda_2^n \sum_{j=0}^{m} (\lambda_1^k\lambda_2^s - \lambda_2^{s+k})^j (\lambda_1^s\lambda_2^k - \lambda_1^k\lambda_2^s)^{m-j}]$
$= \frac{1}{(\lambda_1-\lambda_2)^{m+1}} \{\lambda_1^n (\lambda_1^{s+k} - \lambda_1^k\lambda_2^s)^m - \lambda_2^n (\lambda_1^s\lambda_2^k - \lambda_2^{s+k})^m\}$
$= \frac{1}{(\lambda_1-\lambda_2)^{m+1}} \{(\lambda_1^n\lambda_1^{km} - \lambda_2^n\lambda_2^{km})(\lambda_1^s - \lambda_2^s)^m\}$
$= \frac{1}{(\lambda_1-\lambda_2)^{m+1}} \{(\lambda_1^n\lambda_1^{km} - \lambda_2^n\lambda_2^{km})(\lambda_1^s - \lambda_2^s)^m\}$
$= (\frac{\lambda_1^{km+n} - \lambda_2^{km+n}}{\lambda_1-\lambda_2})(\frac{\lambda_1^s - \lambda_2^s}{\lambda_1-\lambda_2})^m$
$= B_s^m B_{km+n}$.
In order to prove (9), noting that $C_n = B_{n+1} - 3B_n$ and from (8), we obtain
$\sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} B_k^j B_{k-s}^{m-j} C_{js+n}$
$= \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} B_k^j B_{k-s}^{m-j} B_{js+n+1} - 3\sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} B_k^j B_{k-s}^{m-j} B_{js+n}$
$= B_s^m (B_{km+n+1} - 3B_{km+n+1})$
$= B_s^m C_{km+n}$.
In the special case, for $s = 1$ and $n = 0$ in (8), we have

$$B_{km} = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} B_k^j B_{k-1}^{m-j} B_j \tag{10}$$

and the general case $s = 1$ of (8) is

$$B_{km+n} = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} B_k^j B_{k-1}^{m-j} B_{j+n}. \tag{11}$$

Corollary 2.2 *for every integers k, n, we have*

(i) $B_{k-n}B_{k+n} = B_k^2 - B_n^2$
(ii) $B_{2n} = 2B_nC_n, C_{2n} = C_n^2 + 8B_n^2$
(iii) $B_{2n+1} = B_{n+1}^2 - B_n^2$
(iv) $B_k^2 - B_{k-1}Bk + 1 = 1$

Proof. In particular, for $m = 1$, (8) and (9) become

$$B_s B_{k+n} = B_k B_{s+n} - B_{k-s} B_n, \ B_s C_{k+n} = B_k C_{s+n} - B_{k-s} C_n. \qquad (12)$$

Proof of (i) directly follows from (12) by setting $s = k - n$. Substitute $s = 1$, $k = n$ in (9) and using (5), we obtain first part of (ii).
One can prove the second part of (ii) by using (6) and (7) as follows:

$$\begin{aligned} C_{2n} &= B_n C_{n+1} - B_{n-1} C_n \\ &= B_n(3C_n + 8B_n) - B_{n-1} C_n \\ &= C_n(3B_n - B_{n-1} + 8B_n^2 \\ &= C_n^2 + 8B_n^2. \end{aligned}$$

Setting $s = 1, \ k = n + 1$ in (12), we obtain (iii). (iv) is a special case of (1) for $n = 1$.
We observe from (iv) that, B_{k-1} is prime to B_k.
For $m \geq 1$, it follows from (5), that

$$B_s^m B_{km+n} \equiv (-1)^m B_{k-s}^m - B_n + m(-1)^{m-1} B_{k-s}^{m-1} - B_k B_{s+n} \ (mod \ B_k^2). \quad (13)$$

For $s = 1$ in (13), we get

$$B_{km+n} \equiv (-1)^m B_{k-1}^m - B_n + m(-1)^{m-1} B_{k-1}^{m-1} - B_k B_{1+n} \ (mod \ B_k^2), \qquad (14)$$

and therefore for $n = 0$,

$$B_{km} \equiv m(-1)^{m-1} B_{k-1}^{m-1} - B_k \ (mod \ B_k^2). \qquad (15)$$

If (a, b) denote the greatest common divisor of a and b, then from (14) and (15), we observe that

$$(B_{km+n}, B_k) = (B_{k-1}^m, B_k) = (B_k, B_n). \qquad (16)$$

From (16) and Euclidean algorithm, we have the following beautiful result due to Panda[8].

Theorem 2.3 *Let m and n be positive integers. Then*

$$(B_m, B_n) = B_{(m,n)}$$

Corollary 2.4 *If m, n are positive integers, then*

$$B_m | B_n \Leftrightarrow m | n.$$

Proof. By virtue of Theorem 2.2, we have

$$\begin{aligned} m|n &\Leftrightarrow (m, n) = m \\ &\Leftrightarrow B_{(m,n)} = B_m \\ &\Leftrightarrow (B_m, B_n) = B_m \\ &\Leftrightarrow B_m | B_n. \end{aligned}$$

3 Congruences for prime subscripted balancing numbers

Let $(\frac{a}{p})$ be the Legendre symbol of any integer a and any prime p. If p is an odd prime, by quadratic reciprocity law, we have

$$\left(\frac{8}{p}\right) = \left(\frac{p}{8}\right) = \begin{cases} 1, & \text{if } p \equiv 1 \ (mod\ 8) \\ -1, & \text{if } p \equiv 5 \ (mod\ 8). \end{cases}$$

Theorem 3.1 *If p is an odd prime, then*

$$C_p \equiv 3 \ (mod\ p)$$
$$B_p \equiv \left(\frac{p}{8}\right) \ (mod\ p)$$

Proof. Since $\binom{p}{k} k! = p(p-1)\cdots(p-k+1) \equiv 0 \ (mod\ p)$, we observe that

$$p| \binom{p}{k}, k = 1, 2, \cdots (p-1).$$

From this and (4), we get

$$\begin{aligned} C_p &= \frac{1}{2}[(3+\sqrt{8})^p + (3-\sqrt{8})^p] \\ &= \frac{1}{2}[\Sigma_{k=0}^{p} \binom{p}{k} 3^{p-k}(\sqrt{8})^k + (-\sqrt{8})^k] \\ &= \Sigma_{k=0}^{p} \binom{p}{k} 3^{p-k}8^{\frac{k}{2}} \equiv 3 \ (mod\ p). \end{aligned}$$

Similarly using (3) and Euler's criterion and since $(\frac{a}{p}) \equiv a^{p-1} \ (mod\ p)$, we get

$$\begin{aligned} B_p &= \frac{1}{2\sqrt{8}}[(3+\sqrt{8})^p - (3-\sqrt{8})^p] \\ &= \frac{1}{2\sqrt{8}}[\sum_{k=0}^{p} \binom{p}{k} 3^{p-k}\{(\sqrt{8})^k - (\sqrt{8})^{-k}\}] \\ &= \sum_{k=0_{2|k}}^{p} \binom{p}{k} 3^{p-k}8^{\frac{k-1}{2}} \equiv 8^{\frac{p-1}{2}} \equiv \left(\frac{8}{p}\right) = \left(\frac{p}{8}\right) \ (mod\ p). \end{aligned}$$

Theorem 3.2 *Let p be an odd prime. Then*

$$B_{p-1} \equiv 3\left(\left(\frac{p}{8}\right) - 1\right) \ (mod\ p)$$
$$B_{p+1} \equiv 3\left(1 + \left(\frac{p}{8}\right)\right) \ (mod\ p)$$

Proof. From (4), we observe that

$$\begin{aligned} C_p &= B_{p+1} - 3B_p \\ &= 6B_p - B_{p-1} - 3B_p \\ &= 3B_p - B_{p-1} \end{aligned}$$

Thus, $B_{p+1} = 3B_p + C_p$ and $B_{p-1} = 3B_p - C_p$.
These equations together with Theorem 3.1 yield the results.

Corollary 3.3 *Let p be a prime , then $p|B_{p-\left(\frac{p}{8}\right)}$.*

Proof.The proof of the corollary directly follows from Theorem 3.1.

Corollary 3.4 *Let $p > 3$ be a prime and q be a prime divisor of B_p, then*

$$q \equiv \left(\frac{q}{8}\right) \ (mod\ p)$$

Proof. By virtue of Corollary 3.3, since $q|B_{q-\left(\frac{q}{8}\right)}$, we have $q|\left(B_{q-\left(\frac{q}{8}\right)}, B_p\right)$.
By Theorem 2.2, we get $q|B_{\left(p,q-\left(\frac{q}{8}\right)\right)}$ which implies $\left(p, q - \left(\frac{q}{8}\right)\right) = p$ and hence, $q \equiv \left(\frac{q}{8}\right) \ (mod\ p)$.
In[4], it is well known that, for any integer k, $C_k^2 - 8B_k^2 = 1$. Thus the greatest common divisor (C_k, B_k) is either 1 or 2.
Also observe that, for integers k and n with $k \neq 0$, substituting $s = -k$ in (12) and since $B_{2k} = 2B_kC_k$ and using (3), we find

$$B_{k+n} = 2B_nC_k - B_{n-k}. \tag{17}$$

Theorem 3.5 *Let k and n are integers with $k \neq 0$, then*

$$\frac{B_{kn}}{B_k} \equiv \begin{cases} 2m+1 \ (mod\ 8B_k^2), & \text{if } n=2m+1; \\ 2mC_k \ \ (mod\ 8B_k^2), & \text{if } n=2m. \end{cases}$$

Proof. Since $B_{-kn} = -B_{kn}$, it suffices to prove the result for $n \geq 0$. Clearly the theorem holds for $n = 0, n = 1$. Suppose $n \geq 2$ and it holds for all positive integers less than n. For $n = (n-1)k$, (17) becomes

$$B_{kn} = 2B_{(n-1)k}C_k - B_{(n-2)k}$$

and since

$$C_k^2 = 8B_k^2 + 1 \equiv 1 \ (mod\ 8B_k^2),$$

by inductive hypothesis, one can obtain

$$\frac{B_{kn}}{B_k} = 2C_k\frac{B_{(n-1)k}}{B_k} - \frac{B_{(n-2)k}}{B_k}$$

which is either $4mC_k^2 - (2m-1)$ or $2C_k(2m-1) - 2(m-1)C_k$ according as $n = 2m+1$ or $n = 2m$. Thus

$$\frac{B_{kn}}{B_k} \equiv \begin{cases} 2m+1 \ (mod\ 8B_k^2), & \text{if n=2m+1;} \\ 2mC_k \ \ (mod\ 8B_k^2), & \text{if n=2m,} \end{cases}$$

and therefore theorem holds for all positive integers n.

Corollary 3.6 *If p be an odd prime divisor of B_k and k be any nonzero integer, then*

$$\frac{B_{kp}}{B_k} \equiv p \ (mod\ 8p^2)$$

Proof. Since $p|B_k$, $8p^2|8B_k^2$ and by virtue of Theorem 3.3, we get the desired result.

Theorem 3.7 *For positive integers m and n,*

$$B_{mn} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} B_m^i B_{m-1}^{n-i} B_i \equiv 0 \ (mod\ B_m)$$

Proof. Proof of this theorem directly follows from (10).

Corollary 3.8 *For positive integers λ, m, n and prime p, if $p^\lambda \| B_m$, then $p|n$ if and only if $p^{\lambda+1}|B_{mn}$. Where $p^\lambda \| B_m$ means p^λ divides B_m and $p^{\lambda+1}$ does not divide B_m.*

Proof. By Theorem 3.7,

$$B_{mn} \equiv (-1)^{n-1} n B_{m-1}^{n-1} B_m \ (mod\ B_m^2).$$

Since $p|B_m$, $m > 1$, therefore

$$(B_{m-1}, B_m) = B_{(m-1,m)} = 1.$$

Again, since $p^\lambda \| B_m$ and $p^{\lambda+1}|B_m^2$, we get

$$p^{\lambda+1}|B_{mn} \Leftrightarrow p^{\lambda+1}|nB_{m-1}^{n-1}B_m \Leftrightarrow p|n.$$

A very interesting relation among balancing and Lucas-balancing numbers is observed through following theorem.

Theorem 3.9 *For any positive integer m,*

$$B_{2m} \equiv 0 \ (mod\ C_m), B_{2m-1} \equiv 1 \ (mod\ C_m)$$

Proof. The first part of the theorem is obvious as $B_{2m} = 2B_mC_m$.
Since $\lambda_1\lambda_2 = 1$, we observe that

$$\begin{aligned}\lambda_1^{2m-1} - \lambda_2^{2m-1} - (\lambda_1 - \lambda_2) &= \lambda_1^{2m-1} - \lambda_2^{2m-1} - (\lambda_1\lambda_2)^{m-1}(\lambda_1 - \lambda_2)\\ &= (\lambda_1^{m-1} - \lambda_2^{m-1})(\lambda_1^m + \lambda_2^m)\end{aligned}$$

Divide $(\lambda_1 - \lambda_2)$ both sides to get

$$B_{2m-1} = 1 + 2B_{m-1}C_m,$$

which proves second part of the theorem.
We conclude this section by presenting an important remark.

Remark 3.10 *By Corollary 3.8, any prime power divides some positive balancing numbers. Let* $d = p_1^{\lambda_1}p_2^{\lambda_2}\dots p_r^{\lambda_r}$ *where* $p_1 < p_2\dots p_r$ *and suppose* $p_i^{\lambda_i}|B_{n_i}$ *for each* $i = 1, 2, \dots r$. *Since* $B_{n_i}|B_{[n_1,n_2\dots n_k]}$ *and* $p_i^{\lambda_i}|B_{[n_1,n_2\dots n_k]}$, *it follows that* $d|B_{[n_1,n_2\dots n_k]}$. *Thus any positive integer* d *is a divisor of some positive balancing numbers.*

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