Fractional Quaternion Laplace Transform

and Convolution

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Abstract. The concepts of quaternion Fourier transform, quaternion convolution which are based on quaternion algebra have been found to be useful in filter design and color image processing. Here we have proposed the definition of fractional quaternion Laplace transform (FrQLT). Different properties of fractional quaternion Laplace transform are discussed. The convolution theorem in fractional quaternion Laplace transform is proved.

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1. Introduction

The quaternion was invented by the mathematician William Hamilton in 1843. The quaternion can be considered as a four dimensional generalization of a complex number [4]. Scalars, complex and vector signals are special cases of quaternion. Any quaternion may be represented in a hyper complex form as $q = q_r + iq_i + jq_j + kq_k$ where $q_r, q_i, q_j and q_k$ are real’s and $i, j, k,$ obeys the following rules.

$i^2 = j^2 = k^2 = -1 \text{ and } ij = -ji = k, jk = -kj = i, ki = -ik = j$

Recently Guanlei [3] had developed the concept of fractional quaternion Fourier transform. Also Pei and Ding [5] had implemented quaternion Fourier transform for quaternion time-invariant system analysis and filter design.
In full analogy with fractional Fourier transform, Torre [6] introduced fractional Laplace transform as,

\[ L^\alpha (u) = \int_{-\infty}^{\infty} f(x) K^\alpha(x, u) dx, \]

where \( f(x) \) is any square integrable function and \( K^\alpha(x, u) \) is given by

\[ K^\alpha(x, u) = \sqrt{\frac{1 - i \cot \alpha}{2\pi i}} e^{\frac{x^2}{2} \cot \alpha + \frac{u^2}{2} \cot \alpha - xu \csc \alpha}, \alpha \text{ is not multiple of } \pi \]

\[ = \delta(x - u), \quad \alpha \text{ is a multiple of } \pi. \]

In [1, 2] we have discussed some properties of fractional Laplace transform along with its convolution structure.

In this paper we define fractional quaternion Laplace transform and obtained its inverse in section 2. Some properties of fractional quaternion Laplace transform are proved in section 3. Section 4 deals with the convolution of this newly defined transform. Lastly conclusion is given in the fifth section.

2. Fractional quaternion Laplace transform

Definition: For any two dimensional quaternion function \( f(x, y) \) given by

\[ f(x, y) = f_1(x, y) + i f_2(x, y) + j f_3(x, y) + k f_4(x, y) \]

\( f_1(x, y), f_2(x, y), f_3(x, y) \) and \( f_4(x, y) \) are real the fractional quaternion Laplace transform of \( f(x, y) \) is denoted by \( L_{i,j}^{\alpha, \alpha_1}(u, v) \) and

\[ L_{i,j}^{\alpha, \alpha_1}(u, v) = L_{i,j}^{\alpha, \alpha_1}\{f(x,y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{\alpha_1,i}(x,u)f(x,y)k_{\alpha_2,j}(y,v)dx.dy \]

\[ (2.1) \]

where \( k_{\alpha_1,i}(x,u) = \sqrt{\frac{1 - i \cot \phi_1}{2\pi i}} e^{\frac{x^2}{2} \cot \phi_1 + \frac{u^2}{2} \cot \phi_1 - xu \csc \phi_1}, \phi_1 = \alpha_1 \frac{\pi}{2} \]

\[ k_{\alpha_2,j}(x,u) = \sqrt{\frac{1 - i \cot \phi_2}{2\pi j}} e^{\frac{x^2}{2} \cot \phi_2 + \frac{u^2}{2} \cot \phi_2 - xu \csc \phi_2}, \phi_2 = \alpha_2 \frac{\pi}{2} \]

i) If \( \phi_1 = \phi_2 = \frac{\pi}{2} \) i.e. \( \alpha_1 = \alpha_2 = 1 \) definition (2.1) is of quaternion Laplace transform denoted by \( L^0 (u, v) \)

ii) If \( \phi_1 = \frac{\pi}{2}, \phi_2 = 0 \) i.e. \( \alpha_1 = 1, \alpha_2 = 0 \) definition is of one dimensional Laplace transform of \( f(x, y) \) for variable \( x \).

iii) If \( \phi_1 = 0, \phi_2 = \frac{\pi}{2} \) i.e. \( \alpha_1 = 0, \alpha_2 = 1 \) it gives one dimensional Laplace transform of \( f(x, y) \) for variable \( y \)

iv). If \( \phi_1 = 0, \phi_2 = 0 \) i.e. \( \alpha_1 = 0, \alpha_2 = 0 \) definition is equivalent transform of \( f(x, y) \).

v) If \( \phi_1 = \phi_2 = \pi \) i.e. \( \alpha_1 = \alpha_2 = 2 \) then it gives odd-even transform of \( f(x, y) = f(-x, -y) \).
We can now obtain the following relations:

i) \[ L_{i,j}^{0,0} \{ f(x, y) \} = f(x, y). \]
Proof: \[ L_{i,j}^{0,0} \{ f(x, y) \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{0,i}(x, u) f(x, y) k_{0,j}(y, v) \, dx \, dy. \]

\[ = \int_{-\infty}^{\infty} \delta(x, -u) f(x, y) \delta(y, -v) \, dx \, dy = f(x, y). \]

ii) \[ L_{i,j}^{1,1} \{ f(x, y) \} = L_{i,j}^{1,1} (u, v) = L^0 \{ f(x, y) \} = L^0 (u, v). \]

\[ L_{i,j}^{2,2} \{ f(x, y) \} = L_{i,j}^{2,2} (u, v) = P \{ f(x, y) \} = f(-x, -y). \]

iii) \[ L_{i,j}^{3,3} \{ f(x, y) \} = L_{i,j}^{3,3} (u, v) = P \{ L^0 (u, v) \} = L^0 (-u, -v). \]

iv) \[ L_{i,j}^{4,4} \{ f(x, y) \} = L_{i,j}^{4,4} (u, v) = L_{i,j}^{0,0} \{ f(x, y) \} = I \{ f(x, y) \} = f(x, y). \]

Theorem: The quaternion \( f(x, y) \) can be reconstructed from fractional quaternion Laplace transform \( L_{i,j}^{\alpha_1, \alpha_2} \) that is \( L_{i,j}^{\alpha_1, \alpha_2} \{ L_{i,j}^{\alpha_1, \alpha_2} (u, v) \} = f(x, y). \)

Proof: It is clear that for one dimensional function

\[ L^\alpha \{ \delta(t - \tau) \} = L^\alpha \{ u \} = \sqrt{\frac{1 - i \cot \theta_1}{2 \pi i}} e^{\frac{t^2}{2 \cot \theta_1}} e^{\frac{u^2}{2 \cot \theta_1 - u + \csc \theta_1}}, \quad \theta_1 = \alpha \frac{\pi}{2} \neq n \pi, n \in \mathbb{Z}. \]

\[ \delta(x) \times \delta(y) = \]

\[ \therefore L_{i,j}^{\alpha_1, \alpha_2} \{ f_{i,j}^{\alpha_1, \alpha_2} (u, v) \} \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{-\alpha_1,i}(u, s) L_{i,j}^{\alpha_1, \alpha_2} (u, v) k_{-\alpha_2,j}(v, w) \, du \, dv. \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{-\alpha_1,i}(u, s) \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{\alpha_1,i}(x, u) f(x, y) k_{\alpha_2,j}(y, v) k_{\alpha_2,j}(v, w) \, dx \, dy \right) k_{-\alpha_2,j}(v, w) \, du \, dv. \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \left[ \int_{-\infty}^{\infty} k_{-\alpha_1,i}(u, s) k_{\alpha_1,i}(x, u) \, du \right] \left[ \int_{-\infty}^{\infty} k_{-\alpha_2,j}(v, w) k_{\alpha_2,j}(y, v) \, dv \right] \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi i}} \int_{-\infty}^{\infty} e^{(-x)u} f(x, y) \left( \frac{1}{\sqrt{2\pi i}} \int_{-\infty}^{\infty} e^{(w-y)v} \, dv \right) \, dx \right) \, dy \]

\[ = \int_{-\infty}^{\infty} f(x, y) \delta(s - x, w - y) \, dx \, dy \]

\[ = f(x, y) \]
3. Properties of fractional quaternion Laplace transform

**Result 1:** For any quaternion function \( f_n(x,y), n \in N \)
\[
L_{i,j}^{\alpha_1, \alpha_2} \left\{ \sum a_n f_n(x,y) \right\} = \sum a_n L_{i,j}^{\alpha_1, \alpha_2} f_n(x,y)
\]
By linearity property and by using the definition of fractional quaternion Laplace transform, we can prove it easily.

**Result 2:**
\[
L_{i,j}^{\alpha_1, \alpha_2} L_{i,j}^{\alpha_3, \alpha_4} = L_{i,j}^{\alpha_1 + \alpha_3, \alpha_2 + \alpha_4}
\]
**Proof:**
\[
L_{i,j}^{\alpha_3, \alpha_4} L_{i,j}^{\alpha_1, \alpha_2} (f(x,y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{\alpha_3,\alpha_4}(u,v) \left\{ L_{i,j}^{\alpha_1,\alpha_2} (f(x,y)) \right\} k_{\alpha_1,\alpha_2}(u,v) du \, dv
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{\alpha_3,\alpha_4}(u,v) \left\{ \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{\alpha_1,\alpha_2}(x,u) f(x,y) k_{\alpha_2,\alpha_4}(y,v) dx \, dy \right\} k_{\alpha_1,\alpha_2}(u,v) du \, dv \right\}
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{\alpha_3,\alpha_4}(x,y) f(x,y) k_{\alpha_2,\alpha_4}(y,v) dx \, dy
\]
\[
= L_{i,j}^{\alpha_1 + \alpha_3, \alpha_2 + \alpha_4} (f(x,y))
\]
\[
\therefore L_{i,j}^{\alpha_3, \alpha_4} L_{i,j}^{\alpha_1, \alpha_2} = L_{i,j}^{\alpha_1 + \alpha_3, \alpha_2 + \alpha_4}
\]
Similarly, \( L_{i,j}^{\alpha_1, \alpha_2} L_{i,j}^{\alpha_3, \alpha_4} = L_{i,j}^{\alpha_1 + \alpha_3, \alpha_2 + \alpha_4} \)

**Result 3:**
\[
L_{i,j}^{\alpha_1, \alpha_2} L_{i,j}^{\alpha_3, \alpha_4} = L_{i,j}^{\alpha_1 + \alpha_3, \alpha_2 + \alpha_4}
\]
and
\[
L_{i,j}^{\alpha_1, \alpha_2} L_{i,j}^{\alpha_3, \alpha_4} = \left( L_{i,j}^{\alpha_1, \alpha_2} L_{i,j}^{\alpha_3, \alpha_4} \right) L_{i,j}^{\alpha_1, \alpha_2}
\]
**Proof:** It can be proved easily by using property 2.

**Result 4:**
\[
L_{i,j}^{\alpha_1, \alpha_2} L_{i,j}^{\alpha_1, \alpha_2} = L_{i,j}^{\alpha_1, \alpha_2}
\]
It is obvious because \( \phi_1 = \alpha_1 \frac{\pi}{2} \) and \( \phi_2 = \alpha_2 \frac{\pi}{4} \).

if \( (\alpha_1 \pm 4N) \frac{\pi}{2} = \alpha_1 \frac{\pi}{2} + 2\pi N = \phi_1 + 2\pi N \), \( (\alpha_2 \pm 4N) \frac{\pi}{2} = \phi_2 + 2\pi N \)
\[
\therefore L_{i,j}^{\alpha_1, \alpha_2} L_{i,j}^{\alpha_1, \alpha_2} = L_{i,j}^{\alpha_1, \alpha_2}
\]

**Result 5:** If \( L_{i,j}^{\alpha_1, \alpha_2} (f(x,y)) = L_{i,j}^{\alpha_1, \alpha_2} (u, v) \) then
\[
L_{i,j}^{\alpha_1, \alpha_2} (f(-x, -y)) = L_{i,j}^{\alpha_1, \alpha_2} (-u, -v)
\]
**Proof:** The kernel of fractional Laplace transform is
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\[ k_{\alpha}(t, u) = \sqrt{\frac{1 - icot\phi}{2\pi}} e^{\frac{cot\phi}{2}(t^2 + u^2 - 2tu\sec\phi)} = c(\alpha)e^{\alpha(\alpha)[t^2 + u^2 - 2tu\sec\phi]} \]

where \( c(\alpha) = \sqrt{\frac{1 - icot\phi}{2\pi}}, a(\alpha) = \frac{cot\phi}{2}, b(\alpha) = \sec\phi \)

\[ Lift^{\alpha_1,\alpha_2}[f(-x, -y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{\alpha_1,i}(-x, u)f(-x, -y)k_{\alpha_2,j}(-y, v)dx
dy \]

\[ = c(\alpha)e^{a(\alpha)u^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ax^2 - 2xvb(\alpha)f(-x, -y)c(\beta)e^{a(\beta)v^2} e^{ay^2 - 2yvb(\beta)}} dx
dy \]

\[ = c(\alpha)e^{a(\alpha)(-u)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ax^2 - 2x(-u)b(\alpha)f(X, Y)c(\beta)e^{a(\beta)(-v)^2} e^{ay^2 - 2y(-v)b(\beta)}} dX \cdot dY \]

\[ = L_{\alpha_1,\alpha_2}^{i,j}(-u, -v) \cdot \{ \text{taking} (-x) = X \text{ and} (-y) = Y \} \]

Similarly \( L_{\alpha_1,\alpha_2}^{i,j}[f(-x, y)] = L_{\alpha_1,\alpha_2}^{i,j}(-u, v) \) and \( L_{\alpha_1,\alpha_2}^{i,j}[f(x, -y)] = L_{\alpha_1,\alpha_2}^{i,j}(u, -v) \)

4. New convolution structure in fractional quaternion Laplace transform

We will give the new convolution theorem based on the generalized translation and generalized framework of convolution as in [2].

\[ f(t\theta_z) = \int \rho(w) F(w)k(w, t)k^*(w, \tau)du, dv \]

with weight function \( \rho(u, v) = 1 \) and the kernel

\[ f(x\theta_z, y\theta_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(u, v)L_{\alpha_1,\alpha_2}^{i,j}(u, v)k_{\alpha_1,i}^{(u,v,x,y)}k_{\alpha_2,j}^{*(u,v,\tau,\mu)} \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1. L_{\alpha_1,\alpha_2}^{i,j}(u, v) \left\{ k_{\alpha_1,i}^{(x,u)}k_{\alpha_2,j}^{*(x,v)} \right\} \left\{ k_{\alpha_1,i}^{(u,v,\tau,\mu)}k_{\alpha_2,j}^{*(u,v,\tau,\mu)} \right\} dudv \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L_{\alpha_1,\alpha_2}^{i,j}(u, v) \left\{ \sqrt{\frac{1 - icot\phi_1}{2\pi i}} e^{a (\tau^2 + u^2 - 2b\tau u)} \right\} \left\{ \sqrt{\frac{1 + icot\phi_1}{2\pi i}} e^{-a (\tau^2 + u^2 - 2b\tau u)} \right\} \]

\[ \left\{ \sqrt{\frac{1 - icot\phi_2}{2\pi i}} e^{A (\mu^2 + v^2 - 2B\mu v)} \right\} \left\{ \sqrt{\frac{1 + icot\phi_2}{2\pi i}} e^{-A (\mu^2 + v^2 - 2B\mu v)} \right\} dudv \]
where \( \frac{1}{2} \frac{\cot \theta_1}{\pi}, b = \sec \theta_1, A = \frac{1}{2} \frac{\cot \theta_2}{\pi}, B = \sec \theta_2 \)

\[
A_1 B_1 e^{-A(x^2 - t^2)} e^{-A(y^2 - \mu^2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L_{i, j}^{x_1, x_2}(u, v) e^{2a(x-t)u} e^{2AB(y-\mu)v} dudv
\]

where \( A_1 = \sqrt{\frac{1 - i \cot \theta_1}{2 \pi i} \frac{1 + i \cot \theta_1}{2 \pi i}} \) and \( B_1 = \sqrt{\frac{1 - i \cot \theta_2}{2 \pi i} \frac{1 + i \cot \theta_2}{2 \pi i}} \)

**Definition:** Convolution for fractional quaternion Laplace transform:

Let the generalized translation of two dimensional function \( f(x, y) \) be denoted by \( f(x \Theta_t, y \Theta_\mu) \).
Then the convolution of two functions \( f(x, y) \) and \( g(x, y) \) is defined as

\[
Z(x, y) = (f \Theta g)(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau, \mu) g(x \Theta_\tau, y \Theta_\mu) d\tau d\mu
\]

**Theorem:** Let \( z(x, y) = f(x, y) \Theta g(x, y), \) and
\( F_{i,j}^{x_1, x_2}(u, v), G_{i,j}^{x_1, x_2}(u, v), Z_{i,j}^{x_1, x_2}(u, v) \) be the fractional quaternion Laplace transform of \( f(x, y), g(x, y) \) and \( z(x, y) \) respectively, then

\[
Z_{i,j}^{x_1, x_2}(u, v) = F_{i,j}^{x_1, x_2}(u, v) \cdot G_{i,j}^{x_1, x_2}(u, v),
\]

**Proof:**

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{1 - i \cot \phi_1}{2 \pi i} \frac{x^2}{2 \cos \phi_1 + \frac{u^2}{2} \cos \phi_1 - \frac{u \sin \phi_1}{2} \cos \phi_1} \right] \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau, \mu) A_1 B_1 \frac{1}{2 \pi} e^{-A(x^2 - t^2)} e^{-A(y^2 - \mu^2)} \right]
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{i,j}^{x_1, x_2}(u, v) e^{2a(x-t)u} e^{2AB(y-\mu)v} dudv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{1 - i \cot \phi_2}{2 \pi i} \frac{y^2}{2 \cos \phi_2 + \frac{v^2}{2} \cos \phi_2 - \frac{v \sin \phi_2}{2} \cos \phi_2} \right] d\tau d\mu
\]

\[
= \frac{1 - i \cot \phi_1}{2 \pi i} \frac{1 + i \cot \phi_1}{2 \pi i} \frac{1 - i \cot \phi_2}{2 \pi i} \frac{1 + i \cot \phi_2}{2 \pi i} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{i,j}^{x_1, x_2}(\tau, \mu) \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{i,j}^{x_1, x_2}(u, v) dudv
\]
But \( g(x, y) = \int \int \rho(u, v) G_{i,j}^{\infty, 1, \infty, 2}(u, v) k_{\infty, 1,i}(x, u) k_{\infty, 2,j}(y, v) \, du \, dv \), \( \{ \rho(u, v) = 1 \)

\[
= F_{i,j}^{\infty, 1, \infty, 2}(u, v) \sqrt{\frac{1 + icot \theta_1}{2\pi i}} \sqrt{\frac{1 + icot \theta_2}{2\pi i}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-a(x^2 + u^2 - 2bxu)} e^{-A(y^2 + v^2 - 2Byv)} \, dx \, dy
\]

\[
= F_{i,j}^{\infty, 1, \infty, 2}(u, v) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \, k_{\infty, 1,i}(x, u) k_{\infty, 2,j}(y, v) \, dx \, dy
\]

\[
\therefore Z_{i,j}^{\infty, 1, \infty, 2}(u, v) = F_{i,j}^{\infty, 1, \infty, 2}(u, v) G_{i,j}^{\infty, 1, \infty, 2}(u, v)
\]

5. Conclusion

In this paper we have developed the definition of fractional quaternion Laplace transform and the quaternion convolution. The various properties of fractional quaternion Laplace transform are discussed. The main contribution of our paper is that, we have developed convolution theorem in fractional quaternion Laplace transform. The fractional convolution of the product of two quaternion functions is the product of their convolution. The concept of fractional quaternion Laplace transform, quaternion convolution have been found to be useful in image processing.

References


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