On $I$-Convergence of Double Sequences
in 2-Normed Spaces

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Abstract

The concept of $I$-convergence was introduced by Kostyrko et al (2001). It seems therefore reasonable to investigate the concept of $I$-convergence for the double sequences in 2-normed spaces. In this article we define and investigate ideal analogue of convergence for double sequences in 2-normed space and so we extend this concepts to $I_2$-limit points and $I_2$-cluster points in this spaces. We prove some basic properties.

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1. Introduction

The notion of statistical convergence was introduced first by Fast[7]. The idea of $I$-convergence was introduced by Kostyrko et al (2001) and also independently by Nuray and Ruckle (2000), as a generalization of statistical convergence.
Very recently some works on $I$-convergence of double sequences have also been done (see [5, 6, 21, 27]). The concept of linear 2-normed spaces has been investigated by Gačler in 1960's [11, 12] and has been developed extensively in different subjects by others [3, 14, 23, 25]. Note that the notion of the statistical convergence for the double sequences in 2-normed spaces was introduced in papers [24]. It seems therefore reasonable to investigate the concept of $I$-convergence for the double sequences in 2-normed spaces.

Throughout this paper $\mathbb{N}$ will denote the set of positive integers. Let $(X, \| \cdot \|)$ be a normed space. Let $E$ be subset of positive integers $\mathbb{N}$ and $j \in \mathbb{N}$. The quotient $d_j(E) = \frac{\text{card}(E \cap \{1, ..., j\})}{j}$ is called the $j$'th partial density of $K$. Note that $d_j$ is a probability measure on $\mathcal{P}(\mathbb{N})$, with support $\{1, ..., j\}$ [2, 4, 7, 24].

The limit $d(E) = \lim_{j \to \infty} d_j(K)$ is called the natural density of $E \subseteq \mathbb{N}$ (if exists). Clearly, finite subsets have natural density zero and $d(E^c) = 1 - d(E)$ where $E^c = E - \mathbb{N}$, i.e., the complement of $E$ [2, 27].

Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $X$ is said to be statistically convergent to $l \in X$ if the set $A(\epsilon) = \{n \in \mathbb{N} : \|x_n - l\| \geq \epsilon\}$ has natural density zero for each $\epsilon > 0$ in other words for each $\epsilon > 0$,

$$\lim \frac{1}{n} \text{card}\{k \leq n : |x_k - l| \geq \epsilon\} = 0$$

and $x = (x_n)_{n \in \mathbb{N}}$ is called to be statistically Cauchy sequence if for each $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that

$$\lim \frac{1}{n} \text{card}\{k \leq n : |x_k - x_N| \geq \epsilon\} = 0$$

The convergence of a double sequence introduce by many manner [4, 21, 22]. By the convergence of a double sequence we mean the convergence in Pringsheim’s sense [22]. A double sequence $x = (x_{jk})_{j, k \in \mathbb{N}}$ is called to be convergent in the Pringsheim’s sense if for each $\epsilon > 0$ there exist a positive integer $N = N(\epsilon)$ such that for all $j, k \geq N$ implies $|x_{jk} - l| < \epsilon$. $L$ is called the Pringsheim limit of $x$.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a set of positive integers and let $A(n, m)$ be the numbers of $(j, k)$ in $A$ such that $j \leq n$ and $k \leq m$. Then the two-dimensional concept of natural density can be defined as follows.

The lower asymptotic density of a set $A \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$d_2(A) = \liminf_{n, m} \frac{A(n, m)}{nm}$$
If the sequence \((\frac{A(n,m)}{nm})_{n,m\in\mathbb{N}}\) has a limit in Pringsheim’s sense then we say that \(A\) has a *double natural density* and is defined as
\[
d_2(A) = \lim_{n,m} \frac{A(n,m)}{nm}
\]

Next we recall the following definition, where \(Y\) represents an arbitrary set.

**Definition 1.1.** A family \(\mathcal{I} \subseteq \mathcal{P}(Y)\) of subsets a nonempty set \(Y\) is said to be an ideal in \(Y\) if:

i) \(\emptyset \in \mathcal{I}\)

ii) \(A, B \in \mathcal{I}\) implies \(A \cup B \in \mathcal{I}\)

iii) \(A \in \mathcal{I}, B \subseteq A\) implies \(B \in \mathcal{I}\)

\(\mathcal{I}\) is called a nontrivial ideal if \(X \notin \mathcal{I}\).

**Definition 1.2.** Let \(Y \neq \emptyset\). An empty family \(F\) of subsets of \(Y\) is said to be a *filter* in \(Y\) provided:

i) \(\emptyset \in F\).

ii) \(A, B \in F\) implies \(A \cap B \in F\).

iii) \(A \in F, A \subseteq B\) implies \(B \in F\).

If \(\mathcal{I}\) is a nontrivial ideal in \(Y, Y \neq \emptyset\), then the class
\[
F(\mathcal{I}) = \{M \subset Y : (\exists A \in \mathcal{I})(M = Y - A)\}
\]
is a filter on \(Y\), called the *filter associated with* \(\mathcal{I}\).

**Definition 1.3.** A nontrivial ideal \(\mathcal{I}\) in \(Y\) is called *admissible* if \(\{x\} \in \mathcal{I}\) for each \(x \in Y\).

**Definition 1.4.** A nontrivial ideal \(\mathcal{I}\) in \(\mathbb{N} \times \mathbb{N}\) is called *strongly admissible* if \(\{i\} \times \mathbb{N}\) and \(\mathbb{N} \times \{i\}\) belong to \(\mathcal{I}\) for each \(i \in \mathbb{N}\).

It is evident that a strongly admissible ideal is admissible also.

Let \(\mathcal{I}_0=\{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \in \mathbb{N} \times \mathbb{N} - A)\}\).

Then \(\mathcal{I}_0\) is a nontrivial strongly admissible ideal and clearly an ideal \(\mathcal{I}\) is strongly admissible if and only if \(\mathcal{I}_0 \subseteq \mathcal{I}\). [5]

Let \(\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})\) be a nontrivial ideal in \(\mathbb{N}\). The sequence \((x_n)_{n\in\mathbb{N}}\) in \(X\) is said to be \(\mathcal{I}\)-convergent to \(x \in X\), if for each \(\epsilon > 0\) the set \(A(\epsilon)=\{n \in \mathbb{N} : \|x_n - x\| \geq \epsilon\}\) belongs to \(\mathcal{I}\) [1,17,19].

### 2. Preliminary Notes

The concept of \(\mathcal{I}\)-convergence of double sequences in metric spaces \(X\) is defined as follow.
Definition 2.1. A double sequence \( x=(x_{jk})_{j,k\in\mathbb{N}} \) of elements of \( X \) is said to be \( I \)-convergent to \( l \in X \) if for every \( \varepsilon > 0 \) we have \( A(\varepsilon) \in I \), where \( A(\varepsilon)=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\rho(x_{mn},l)\geq\varepsilon\} \) and we write it as

\[
I-\lim_{m,n} x_{mn} = l
\]

The notion of linear 2-normed spaces has been investigated by Gähler in 1960's [11,12] and has been developed extensively in different subjects by others [3,14,23].

Let \( X \) be a real linear space of dimension greater than 1, and \( \|\cdot,\cdot\| \) be a non-negative real-valued function on \( X \times X \) satisfying the following conditions:

G1) \( \|x, y\| = 0 \) if and only if \( x \) and \( y \) are linearly dependent vectors.
G2) \( \|x, y\| = \|y, x\| \) for all \( x, y \) in \( X \).
G3) \( \|\alpha x, y\| = |\alpha|\|x, y\| \) where \( \alpha \) is real
G4) \( \|x + y, z\| \leq \|x, z\| + \|y, z\| \) for all \( x, y, z \) in \( X \)

\( \|\cdot,\cdot\| \) is called a 2-norm on \( X \) and the pair \( (X,\|\cdot,\cdot\|) \) is called a linear 2-normed space. In addition, for all scalars \( \alpha \) and all \( x, y, z \) in \( X \), we have the following properties:

1) \( \|\cdot,\cdot\| \) is nonnegative.
2) \( \|x, y\| = \|x + \alpha x\| \)
3) \( \|x - y, y - z\| = \|x - y, x - z\| \)

Some of the basic properties of 2-norm introduce in [23].

As an example of a 2-normed space we may take \( X = \mathbb{R}^2 \) being equipped with the 2-norm \( \|x, y\| := \text{the area of the parallelogram spanned by the vectors } x \) and \( y \), which may be given clearly by the formula

\[
(2.1) \quad \|x, y\| = |x_1y_2 - x_2y_1|, \quad x = (x_1, x_2) \quad y = (y_1, y_2)
\]

Given a 2-normed space \( (X,\|\cdot,\cdot\|) \), one can derive a topology for it via the following definition of the limit of a sequence: a sequence \( (x_n)_{n\in\mathbb{N}} \) in \( X \) is said to be convergent to \( x \) in \( X \) if \( \lim_{n\to\infty} \|x_n - x, z\| = 0 \) for every \( z \in X \). This can be written by the formula:

\[
(\forall z \in Y)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \quad \|x_n - x, z\| < \varepsilon
\]

We write it as

\[
x_n \xrightarrow{\|\cdot,\cdot\|_X} x
\]

Definition 2.2. A sequence \( (x_n)_{n\in\mathbb{N}} \) is a Cauchy sequence in a 2-normed space \( (X,\|\cdot,\cdot\|) \) if \( \lim_{n,m} \|x_n - x_m, z\| = 0 \) for every \( z \in X \).

Recall that \( (X,\|\cdot,\cdot\|) \) is a 2-Banach space, if every Cauchy sequence in \( X \) is convergence to some \( x \in X \).
Definition 2.3. Let $(X, \|\cdot\|)$ be 2-normed space and $x \in X$. We say that $x$ is an accumulation point of $X$ if there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of distinct elements of $X$ such that $x_k \neq x$ (for any $k$) and $x_n \xrightarrow{\|\cdot\|} x$.

Lemma 2.4. [15] Let $v = \{v_1, ..., v_k\}$ be a basis of $X$. A sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ is convergent to $x$ in $X$ if and only if $\lim_{n \to \infty} \|x_n - x, v_i\| = 0$ for every $i = 1, ..., k$. We can define the norm $\|\cdot\|_\infty$ on $X$ by

$$\|x\|_\infty := \max\{\|x, v_i\| : i = 1, ..., d = k\}$$

Associated to the derived norm $\|\cdot\|_\infty$, we can define the (open) balls $B_{v_1, v_2, ..., v_n}(x, r) = B_v(x, r)$ centered at $x$ having radius $r$ by

$$B_v(x, r) := \{y : \|x - y\|_\infty < r\}$$

Lemma 2.5. [15] A sequence $(x_n)_{n \in \mathbb{N}}$ in $X$ is convergent to $x$ in $X$ if and only if $\lim_{n \to \infty} \|x_n - x\|_\infty = 0$.

Example 2.6. Let $X = \mathbb{R}^2$ be equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram spanned by the vectors $x$ and $y$, which may be given explicitly by the formula

$$\|x, y\| = |x_1y_2 - x_2y_1|, \quad x = (x_1, x_2), \quad y = (y_1, y_2)$$

Take the standard basis $\{i, j\}$ for $\mathbb{R}^2$.

Then, $\|x, i\| = |x_2|$ and $\|x, j\| = |x_1|$, and so the derived norm $\|\cdot\|_\infty$ with respect to $\{i, j\}$ is

$$\|x\|_\infty = \max\{|x_1|, |x_1|\}, \quad x = (x_1, x_2)$$

Thus, here the derived norm $\|\cdot\|_\infty$ is exactly the same as the uniform norm on $\mathbb{R}^2$. Since the derived norm is norm, it is equivalent to Euclidean norm on $\mathbb{R}^2$.

3. $I_2$-LIMIT POINTS AND $I_2$-CLUSTER POINTS IN 2-NORMED SPACES

In [13], the concepts of an ordinary limit points and $I$-limit points for a single sequences was generalized in 2-normed spaces. In this section, we define $I_2$-convergence for double sequence in 2-normed spaces and so we extend this concepts to $I_2$-limit points and $I_2$-cluster points in this spaces. For the following definition we were inspired by Pringsheims [22].

Definition 3.1. Let $x=(x_{jk})_{j,k \in \mathbb{N}}$ be a double sequence in 2-normed space $(X, \|\cdot\|)$. A double sequence $x=(x_{jk})_{j,k \in \mathbb{N}}$ is said to be convergent to $l \in X$ if
\[(\forall z \in X)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall j, k \geq N) \|x_{jk} - l, z\| < \varepsilon\]

We write it as

\[x_{jk} \xrightarrow{\| \cdot \|_X, l} l\]

A double sequence \(x=(x_{jk})_{j,k \in \mathbb{N}}\) is said to be bounded if for each nonzero \(z \in X\) and for each \(j, k \in \mathbb{N}\) there exists \(M > 0\) such that \(\|x_{jk}, z\| < M\).

Note that a convergent double sequence need not be bounded.

Now we define the \(I_2\) and \(I_2^*\)-convergence for double sequence \(x=(x_{jk})_{j,k \in \mathbb{N}}\) as follows:

**Definition 3.2.** A double sequence \(x=(x_{jk})_{j,k \in \mathbb{N}}\) in 2-normed space \((X, \|\cdot, \cdot\|)\) is said to be \(I_2\)-convergence to \(l \in X\) if for all \(\varepsilon > 0\) and nonzero \(z \in X\) the set

\[A(\varepsilon) = \{(j,k): \|x_{jk} - l, z\| \geq \varepsilon\} \in I_2\]

In this case we write it as

\[I_2 - \lim_{j,k} x_{jk} = l\]

**Remark 3.3.** Put \(I_d = \{A \subset \mathbb{N} \times \mathbb{N} : d_2(A) = 0\}\). Then \(I_d\) is an admissible ideal in \(\mathbb{N} \times \mathbb{N}\) and \(I_{d_2}\)-convergence becomes statistical convergence\[24\].

**Remark 3.4.** Not that if \(I\) is the ideal \(I_0\) then \(I_2\)-convergence coincide with the usual convergence(Definition 3.1).

**Remark 3.5.** If \(x=(x_{jk})_{j,k \in \mathbb{N}}\) is \(I_2\)-convergent, then \((x_{jk})_{j,k \in \mathbb{N}}\) need not be convergent. Also it is not necessarily bounded. This actuality can be seen from the next example.

**Example 3.6.** Let \((X, \|\cdot, \cdot\|)\) be 2-normed space introduced in Example 2.6 and the \(x=(x_{jk})_{j,k \in \mathbb{N}}\) be defined as

\[(1,1) \text{ otherwise}\]

and let \(l = (1,1)\).

Then for every \(\varepsilon > 0\) and \(z \in X\)

\[\{(j,k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\} \subseteq \{1, 4, 9, 16, \ldots, j^2, \ldots\} \times \{1, 4, 9, 16, \ldots, k^2, \ldots\} \times \{1, 4, 9, 16, \ldots, k^2, \ldots\} \times \{1, 4, 9, 16, \ldots, k^2, \ldots\} \times \ldots\].

Hence

the cardinality of the set \(\{(j,k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\} \leq \sqrt{j} \sqrt{k}\) for
each $\varepsilon > 0$.
This implies $d_2(\{(j,k), j \leq n, k \leq m : \|x_{jk} - l, z\| \geq \varepsilon\}) = 0$ for each $\varepsilon > 0$ and $z \in X$. We have
\[
I_{d_2} - \lim_{j,k} x_{jk} = l
\]

But $x = (x_{jk})_{j,k \in \mathbb{N}}$ is neither convergent to $l$ nor bounded.

Remark 3.7. The following corollary can be verified that if $x = (x_{jk})_{j,k \in \mathbb{N}}$ be $I_2$-convergent to $l \in X$, then $l$ is determined uniquely.

**Corollary 3.8.** Let $x = (x_{jk})_{j,k \in \mathbb{N}}$ be a convergent double sequence in 2-normed space $(X, \| \cdot \|)$ and $l_1, l_2 \in X$. If $I_2\lim_{j,k} \|x_{jk} - l_1, z\| = 0$ and $I_2\lim_{j,k} \|x_{jk} - l_2, z\| = 0$ then $l_1 = l_2$.

**Proof:** Let $l_1 \neq l_2$, hence there exists $z \in X$ such that $0 \neq l_1 - l_2$ and $z$ are linearly independent. Put
\[
\|l_1 - l_2, z\| = 2\varepsilon, \text{ with } \varepsilon > 0
\]
Now
\[
2\varepsilon = \|l_1 - x_{jk} + x_{jk} - l_2, z\| \leq \|x_{jk} - l_1, z\| + \|x_{jk} - l_2, z\|
\]
Therefor
\[
\{(j,k) : \|x_{jk} - l_2, z\| < \varepsilon\} \subseteq \{(j,k) : \|x_{jk} - l_1, z\| \geq \varepsilon\} \in I
\]
Hence $\{(j,k) : \|x_{jk} - l_2, z\| < \varepsilon\} \in I$ that is contradict with nontrivial $I$.

**Corollary 3.9.** If $(x_{jk})_{j,k \in \mathbb{N}}, (y_{jk})_{j,k \in \mathbb{N}}$ be double sequences in 2-normed space $(X, \| \cdot \|)$ and $I_2\lim_{j,k} x_{jk} = a, I_2\lim_{j,k} y_{jk} = b$ then
(i) $I_2\lim_{j,k} x_{jk} + y_{jk} = a + b$
(ii) $I_2\lim_{j,k} \alpha x_{jk} = \alpha a$, where $\alpha \in \mathbb{R}$

**Proof:** (i) Let $\varepsilon > 0$. For each nonzero $z \in X$ we have
\[
\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|(x_{mn} + y_{mn}) - (a + b), z\| \geq \varepsilon\} \subseteq \left(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - a, z\| \geq \frac{\varepsilon}{2}\right)
\]
\[
\cup \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|y_{mn} - b, z\| \geq \frac{\varepsilon}{2}\}\right) \subseteq I
\]
Hence $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|(x_{mn} + y_{mn}) - (a + b), z\| \geq \varepsilon\} \in I$ and the statements is follows.
(ii) The statement is an easy consequence of (i)
Definition 3.10. Let $K \subseteq \mathbb{N} \times \mathbb{N}$ such that for each $(j, k) \in \mathbb{N} \times \mathbb{N}$ there exists $(m, n) \in K$ such that $(m, n) > (j, k)$ with respect to the dictionary ordering. If $x = (x_{jk})_{j,k \in \mathbb{N}}$ is a double sequence in $X$, then we call $(x)_K = \{x_{mn} : (m, n) \in K\}$ as a subsequence of $(x_{jk})_{j,k \in \mathbb{N}}$.

Definition 3.11. An element $l \in X$ is said to be limit point of a double sequence $(x_{jk})_{j,k \in \mathbb{N}}$ in 2-normed space $(X, \|\cdot, \cdot\|)$ if there exists a subsequence of $x$ which is convergent to $l$.

Example 3.12. Let $(X, \|\cdot, \cdot\|)$ be 2-normed space introduced in Example 3.6 and the $x=(x_{jk})_{j,k \in \mathbb{N}}$ be defined as

$$(1,k) \text{ otherwise}$$

and let $l = (1, 1)$.

We put $K = \{(m, m) : m \in \mathbb{N}\}$. Then $(x)_K$ is subsequence of $x = (x_{jk})_{j,k \in \mathbb{N}}$ and $l = (1, 1)$ is a limit point of a double sequence $(x_{jk})_{j,k \in \mathbb{N}}$.

Definition 3.13. Let $x = (x_{jk})_{j,k \in \mathbb{N}}$ be a double sequence in 2-normed space $(X, \|\cdot, \cdot\|)$. An element $l \in X$ is said to be an $I_2$-limit point of $(x_{jk})_{j,k \in \mathbb{N}}$ if there exists a set $M = \{(m_j, m_k) : j, k \in \mathbb{N}\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $M \notin I$ and $\lim_{m_j,m_k} x_{m_j,m_k} = l$

We now introduce the notations $L^2_x$ and $I(\Lambda^2_x)$ to denote the set of all limit points and $I$-limit points of $(x_{jk})_{j,k \in \mathbb{N}}$ respectively.

Example 3.14. If we consider double sequence $(x_{jk})_{j,k \in \mathbb{N}}$ introduced in Example 3.12 and $I = \{A \subset \mathbb{N} \times \mathbb{N} : d_2(A) = 0\}$ then $I(\Lambda^2_x) = \emptyset$. Otherwise, there exists $M = \{(m_j, m_k) \in \mathbb{N} \times \mathbb{N} : j, k \in \mathbb{N}\}$ such that $d_2(M) > 0$ By definition $x = (x_{jk})_{j,k \in \mathbb{N}}$ we have $m_j \neq m_k \Rightarrow x_{m_j,m_k} = (1, m_k)$ and

$$\|x_{m_j,m_k} - \beta\|_\infty = \max\{|1 - b_1|, |m_k - b_2|\} \text{ where } \beta = (b_1, b_2).$$

If $m_j = m_k \Rightarrow x_{m_j,m_k} = (1, 1)$ and

$$\|x_{m_j,m_k} - \beta\|_\infty = \max\{|1 - b_1|, |1 - b_2|\}$$
On another word \( d_2(M) = \lim_{m,n} M(m,n) > 0 \)
Hence \( \lim \| x_{m,n} \beta \|_\infty \neq 0 \).

We show in Example 3.12 and 3.14 that in general, \( L^2_\alpha \) and \( I(\Lambda^2_\alpha) \) may be quit deferent.

**Definition 3.15.** An element \( \alpha \in X \) is said to be an \( I_2 \)-cluster point of a
double sequence \( x = (x_{jk})_{j,k \in \mathbb{N}} \) in 2-normed space \( (X, \| ., . \|) \) if for each \( \varepsilon > 0 \)
and nonzero \( z \in X \) the set \( \{ (j, k) : \| x_{jk} - \alpha, z \| < \varepsilon \} \notin I \).
We denote the set of all \( I_2 \)-cluster points of \( x \) by \( I(\Gamma^2_\alpha) \).

**Theorem 3.16:** Let \( I \) be a strongly admissible ideal. Then for any double
sequence \( x = (x_{jk})_{j,k \in \mathbb{N}} \) in \( (X, \| ., . \|) \) we have \( I(\Lambda^2_\alpha) \subseteq I(\Gamma^2_\alpha) \).

**Proof:** Let \( \alpha \in I(\Lambda^2_\alpha) \). Then there exists a set
\( M = \{ (m_i, m_j) \in \mathbb{N} \times \mathbb{N} : j, k \in \mathbb{N} \} \)
such that
\[
\lim_{m_j,m_k} \| x_{m_j,m_k} - \alpha, z \| = 0 \text{ for each } z \in X. \tag{1}
\]
Let \( \varepsilon > 0 \). By (1) there exists \( n_0 \in \mathbb{N} \) such that:
for all \( m_j, m_k \geq n_0 \) we have \( \| x_{m_j,m_k} - \alpha, z \| < \varepsilon \) for each nonzero \( z \in X \).
So, for each nonzero \( z \in X \) we have
\[
\{ (j, k) : \| x_{jk} - \alpha, z \| < \varepsilon \} \supseteq M \setminus \{ (m_j, m_k) : \text{either } m_j \leq n_0 - 1 \text{ or } m_k \leq n_0 - 1 \}. 
\]
Since \( I \) is strongly admissible, so
\[
\{ (j, k) : \| x_{jk} - \alpha, z \| < \varepsilon \} \notin I \text{ for each } z \in X
\]
This implies \( \alpha \in I(\Gamma^2_\alpha) \), which completes the proof.

**Corollary 3.17:** Let \( (X, \| ., . \|) \) be finite dimensional 2-normed space and
\( I \subseteq \mathbb{N} \times \mathbb{N} \) be a admissible ideal. Then for each double sequence \( x = (x_{jk})_{j,k \in \mathbb{N}} \)
in \( (X, \| ., . \|) \) the set \( I(\Gamma^2_\alpha) \) is closed in \( X \).

**Proof:** Let \( y \in \overline{I(\Gamma^2_\alpha)} \). Put \( \varepsilon > 0 \) then there exists \( l \in I(\Gamma^2_\alpha) \cap B(y, \varepsilon) \). Choose
\( \delta > 0 \) such that \( B_v(l, \delta) \subseteq B_v(y, \varepsilon) \). Hence we have
\[
\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \| y - x_{mn}, z \| < \varepsilon \} \supseteq \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \| l - x_{mn}, z \| < \delta \} \notin I
\]
Therefor \( \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \| y - x_{mn}, z \| < \varepsilon \} \notin I \) and \( y \in I(\Gamma^2_\alpha) \).

**Corollary 3.18:** Let \( (X, \| ., . \|) \) be 2-normed space and \( M^2_\alpha \) be the set of all
bounded double sequence of \( X \) with norm
\[
\| x \| = sup_{m,n} \| x_{mn}, z \| \text{ for each } z \in X, \text{ where } x = (x_{mn})_{m,n \in \mathbb{N}}
\]
Then $\mathcal{M}_2^2$ is a norm linear space.

**Theorem 3.19:** Let $(X, \| \cdot \|)$ be a 2-Banach space. If $I$ be a nontrivial admissible ideal of $\mathbb{N} \times \mathbb{N}$ and $\mathcal{M}_2^I$ denoted the set all bounded $I_2$-convergent double sequences of $X$ then the set $\mathcal{M}_2^I$ is a closed linear subspace of the norm linear space $\mathcal{M}_2^2$.

**proof:** From Corollary (3.9) we see that $\mathcal{M}_2^I$ is a linear subspace of $\mathcal{M}_2^2$. Therefore we only show that $\mathcal{M}_2^I$ is closed in $\mathcal{M}_2^2$.

Let $x^p \in \mathcal{M}_2^I (p = 1, 2, ...)$ and $\lim_p \| x^p - x \| = 0$ for each $z \in X$ and $x \in \mathcal{M}_2^2$. We claim that $x \in \mathcal{M}_2^I$.

Since $x^p \in \mathcal{M}_2^I$ for each $p$ there exists an element $a_p \in X$ such that

$$I_2\text{-lim}_{m,n} x^p_{mn} = a_p (p = 1, 2, ...) ,$$

where $x^p = (x_{mn})_{m,n\in\mathbb{N}}$.

We now prove the following statements:

(i) There exists $a \in X$ such that $a_p \xrightarrow{I_2\text{-x}} a$.

(ii) $I_2\text{-lim}_{m,n} x_{mn} = a$, where $x = (x_{mn})_{m,n\in\mathbb{N}}$.

The result will then follows from (i) and (ii).

**Proof of (i):**

We have $x^p \xrightarrow{\| \cdot \|_X} x \in m_2$. Hence, for each $\varepsilon > 0$ and $z \in X$, there exists $n_0 \in \mathbb{N}$ such that, for each $q \geq r \geq N_0$, we have,

$$\| x^q - x^r, z \| < \frac{\varepsilon}{3}$$

Now since $x^q, x^r \in m_2^I$, so $I_2\text{-lim}_{m,n} x^q_{mn} = a_q$ and $I_2\text{-lim}_{m,n} x^r_{mn} = a_r$. Therefore

$$A_q = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \| x^q - a_q, z \| < \frac{\varepsilon}{3} \} \in F(I) \text{ for each } z \in X
$$

$$A_r = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \| x^r - a_r, z \| < \frac{\varepsilon}{3} \} \in F(I) \text{ for each } z \in X
$$

Then $A_r \cap A_q \in F(I)$. Since $I$ is nontrivial and admissible so $A_r \cap A_q$ must be infinite set. Choose $(m_0, n_0) \in A_r \cap A_q$ and therefore, for each $z \in X$

$$\| x^q_{m_0n_0} - a_q, z \| < \frac{\varepsilon}{3} \text{ and } \| x^r_{m_0n_0} - a_r, z \| < \frac{\varepsilon}{3}$$

Hence for each $z \in X$ and $q \geq r \geq N_0$ we have,

$$\| a_q - a_r, z \| \leq \| a_q - x^q_{m_0n_0}, z \| + \| x^q_{m_0n_0} - x^r_{m_0n_0}, z \| + \| x^r_{m_0n_0} - a_r, z \| < \varepsilon$$

So $(a_p)_{p \in \mathbb{N}}$ is a Cauchy sequence in 2-Banach space $X$ and so it must converge to an element $a \in X$. Hence $a_p \xrightarrow{\| \cdot \|_X} a$.

**Proof of (ii):**

Let $\delta > 0$. Since $x^p \xrightarrow{\| \cdot \|_X} x$ there exists $q \in \mathbb{N}$ such that

$$\| x^q - x, z \| < \frac{\delta}{3} \text{ for each } z \in X$$

(3.1)
The number \( q \) can be chosen in such a way that together with (3.1) the inequality \( \|a_q - a, z\| < \frac{\delta}{3} \) for each \( z \in X \) also holds.

Because \( I_2\)-\( \lim_{m,n} x_{mn}^{(q)} = a_q \), hence

\[
A_q = \{ (m, n) \in \mathbb{N} : \|x_{mn} - a_q, z\| < \frac{\delta}{3} \} \in F(I) \text{ for each } z \in X.
\]

Now for each \( (m, n) \in A_q \) we have:

\[
\|x_{mn} - a, z\| \leq \|x_{mn} - x_{mn}^{(q)}, z\| + \|x_{mn}^{(q)} - a_q, z\| + \|a_q - a, z\| < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} < \delta.
\]

Hence

\[
A_q = \{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - a, z\| < \delta \} \in F(I) \text{ for each } z \in X.
\]

This implies that:

\[
\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - a, z\| \geq \delta \} \in I \text{ for each } z \in X.
\]

Therefore \( I_2\)-\( \lim_{m,n} x_{mn} = a \). This completes the proof of the theorem.

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**REFERENCES**


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