On the Hypersurface of a Second Approximate Matsumoto Metric $\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$

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Abstract

The purpose of the present paper is to investigate the various kinds of hypersurfaces of Finsler space with special $(\alpha, \beta)$ metric $\alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}$. We have proved the conditions for this hypersurface to be a hyperplane of 1$^{st}$ kind, 2$^{nd}$ kind and also that this hypersurface is not a hyperplane of 3$^{rd}$ kind.

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1 Introduction

We consider an n-dimensional Finsler space $F^n = (M^n, L)$, i.e., a pair consisting of an n-dimensional differential manifold $M^n$ equipped with a fundamental function $L(x, y)$. The concept of the $(\alpha, \beta)$-metric $L(\alpha, \beta)$ was introduced by M. Matsumoto ([5]) and has been studied by many authors ([1], [2], [8]). As well known example, there are Randers metric $\alpha + \beta$, Kropina metric $\alpha^2/\beta$ and generalised Kropina metric $\alpha^{m+1}/\beta^m (m \neq 0, -1)$ whose studies have greatly contributed to the growth of Finsler geometry. A Finsler metric $L(x, y)$ is called an $(\alpha, \beta)$-metric $L(\alpha, \beta)$ if $L$ is a positively homogeneous function of $\alpha$ and $\beta$ of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M^n$.

A hypersurface $M^{n-1}$ of the $M^n$ may be represented parametrically by the equation $x^i = x^i(u^\alpha), \alpha = 1, ..., n - 1$, where $u^\alpha$ are Gaussian coordinates on $M^{n-1}$.

In the present paper, we consider an n-dimensional Finsler space $F^n =$
(M^n, L) with (α, β)-metric \( L(\alpha, \beta) = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2} \) and the hypersurface of \( F^n \) with \( b_i(x) = \partial_i b \) being the gradient of a scalar function \( b(x) \). We prove the conditions for this hypersurface to be a hyperplane of 1st kind, 2nd kind and we also prove that this hypersurface is not a hyperplane of 3rd kind.

2 Preliminaries

We are devoted to a special Finsler space \( F^n = (M^n, L) \) with the metric

\[ L(\alpha, \beta) = \alpha + \beta + \frac{\beta^2}{\alpha} + \frac{\beta^3}{\alpha^2}. \]  

The derivatives of the (1) with respect to \( \alpha \) and \( \beta \) are given by

\[ L_\alpha = \frac{\alpha^3 - \alpha\beta^2 - 2\beta^3}{\alpha^3}, \quad L_\beta = \frac{\alpha^2 + 2\alpha\beta + 3\beta^2}{\alpha^2}, \]

\[ L_{\alpha\alpha} = \frac{2\beta^2(\alpha + 3\beta)}{\alpha^4}, \quad L_{\beta\beta} = \frac{2(\alpha + 3\beta)}{\alpha^2}, \quad L_{\alpha\beta} = \frac{-2\beta(\alpha + 3\beta)}{\alpha^2}, \]

where \( L_\alpha = \frac{\partial L}{\partial \alpha}, L_\beta = \frac{\partial L}{\partial \beta}, L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}, L_{\beta\beta} = \frac{\partial^2 L}{\partial \beta^2} \) and \( L_{\alpha\beta} = \frac{\partial^2 L}{\partial \alpha \partial \beta} \).

In the special Finsler space \( F^n = (M^n, L) \) the normalized element of support \( l_i = \partial_i L \) and the angular metric tensor \( h_{ij} \) are given by [7]:

\[ l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i, \]

\[ h_{ij} = pa_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j, \]

where

\[ Y_i = a_{ij} y^j, \]

\[ p = LL_\alpha \alpha^{-1} = \frac{(\alpha + \beta)(\alpha^2 + \beta^2)(\alpha^3 - \alpha\beta^2 - 2\beta^3)}{\alpha^6}, \]

\[ q_0 = LL_{\beta\beta} = \frac{2(\alpha + \beta)(\alpha + 3\beta)(\alpha^2 + \beta^2)}{\alpha^4}, \]

\[ q_1 = LL_{\alpha\beta} \alpha^{-1} = \frac{-2\beta(\alpha + \beta)(\alpha + 3\beta)(\alpha^2 + \beta^2)}{\alpha^6}, \]

\[ q_2 = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}) = \frac{(\alpha + \beta)(\alpha^2 + \beta^2)(3\beta^2\alpha + 8\beta^3 - \alpha^3)}{\alpha^8}. \]

The fundamental tensor \( g_{ij} = \frac{1}{2} \hat{\partial}_i \hat{\partial}_j L^2 \) and it’s reciprocal tensor \( g^{ij} \) is given by [7]

\[ g_{ij} = pa_{ij} + p_0 b_i b_j + p_1 (b_i Y_j + b_j Y_i) + p_2 Y_i Y_j, \]
where
\[ p_0 = q_0 + L_β^2 = \frac{3\alpha^4 + 15\beta^4 + 12\alpha^3\beta + 18\alpha^2\beta^2 + 20\alpha\beta^3}{\alpha^4}, \]
\[ p_1 = q_1 + L^{-1}pL_β = -\frac{6\alpha^3\beta^2 - 12\alpha^2\beta^3 - 15\alpha\beta^4 - 12\beta^5 + \alpha^5}{\alpha^6}, \] (7)
\[ p_2 = q_2 + p^2L^{-2} = \frac{\beta(6\alpha^3\beta^2 + 12\alpha^2\beta^3 - \alpha^5 + 12\alpha\beta^4 + 3\alpha^3\beta^3 + 12\beta^5)}{\alpha^8}. \]

Let \( C_{ijk} = \Gamma^{\alpha}_{ij} + (\Gamma^\beta_{ij})_\alpha \) be the Cartan connection of the special Finsler space \( F^m \). The difference tensor \( D^i_{jk} = \Gamma^{\alpha}_{ij} - \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \) of the special Finsler space \( F^m \) is given by
\[
D^i_{jk} = B^iE_{jk} + F^i_{jk}B_j + F^i_{jk}B_k + B^i_jb_{ok} + B^i_kb_{oj} - b_{om}g^{im}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j + C_{jkm}A^m_s g^{is} + \lambda^s (C^i_{jm}C^m_s + C^i_{km}C^m_s - C^m_{jkm}C^m_{is}). \] (13)
where
\[
B_k = p_0b_k + p_1Y_k, \quad B^i = g^{ij}B_j, \quad F^k_i = g^{kj}F_{ji}.
\]
\[
B_{ij} = \left\{ p_1(a_{ij} - \alpha^{-2}Y_iY_j) + \frac{\partial p_0}{\partial \beta}m_im_j \right\} / 2,
\]
\[
B^k_i = g^{kj}B_{ji}, \quad (14)
\]
\[
\begin{align*}
A^m_k & = B^m_kE_{00} + B^m_kE_{0k} + B_kF^m_0 + B_0F^m_k, \\
\lambda^m & = B^mE_{00} + 2B_0F^m_0, \quad B_0 = B_iy^i.
\end{align*}
\]

where ‘0’ denote contraction with $y^i$ except for the quantities $p_0, q_0$ and $S_0$.

### 3 Induced Cartan connection

Let $F^{n-1}$ be a hypersurface of $F^n$ given by the equations $x^i = x^i(u^\alpha)$. The element of support $y^i$ of $F^n$ is to be taken tangential to $F^{n-1}$, that is
\[
y^i = B^i_\alpha(u)v^\alpha. \quad (15)
\]

The metric tensor $g_{\alpha\beta}$ and HV-torsion tensor $C_{\alpha\beta\gamma}$ of $F^{n-1}$ are given by
\[
g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta, \quad C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma. \quad (16)
\]

At each point $u^\alpha$ of $F^{n-1}$, a unit normal vector $N^i(u, v)$ is defined by
\[
g_{ij}(x(u, v), y(u, v))B^i_\alpha N^j = 0, \quad g_{ij}(x(u, v), y(u, v))N^iN^j = 1. \quad (17)
\]

As for the angular metric tensor $h_{ij}$, we have
\[
h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \quad h_{ij}B^i_\alpha N^j = 0, \quad h_{ij}N^iN^j = 1. \quad (18)
\]

If $(B^\alpha_i, N_i)$ denote the inverse of $(B^i_\alpha, N^i)$, then we have
\[
B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \quad B^\alpha_i B^\beta_i = \delta^\alpha_\beta, \quad (19)
\]
\[
B^\alpha_i N^i = 0, \quad B^i_\alpha N_i = 0, \quad N_i = g_{ij}N^j, \quad (20)
\]
\[
B^k_i = g^{kj}B_{ji}, \quad (21)
\]
\[
B^\alpha_i B^\beta_j + N^iN_j = \delta^\beta_j. \quad (22)
\]

The induced connection $ICT = (\Gamma^*_{\beta\gamma}^\alpha, C^*_\beta^\alpha, C^*_\alpha\beta)$ of $F^{n-1}$ induced from the Cartan’s connection $CT = (\Gamma_{j\beta}^i, \Gamma_{0k}^i, C_{j\beta}^i)$ is given by([6])
\[
\Gamma^*_{\beta\gamma}^\alpha = B^\alpha_i(B^i_{\beta\gamma} + \Gamma_{j\beta}^i B^j_{\gamma}) + M^\alpha_{\beta\gamma}, \quad (23)
\]
\[
C^*_\beta^\alpha = B^\alpha_i(B^i_{0\beta} + \Gamma_{0j}^i B^j_{\beta}), \quad (24)
\]
\[
C^*_\alpha\beta = B^\alpha_iC_{j\beta}^i B^j_{\gamma}. \quad (25)
\]
where

\[ M_{\beta\gamma} = N_i C_{jk}^i B_{\beta}^j B_{\gamma}^k, \quad M_{\beta}^\alpha = g^\alpha\gamma M_{\beta\gamma}, \quad (24) \]

\[ H_\beta = N_i (B_{0\beta}^i + \Gamma^*_{0j} B_{j\beta}^i), \quad (25) \]

and \( B_{\beta\gamma}^i = \partial B_{\beta\gamma}^i / \partial u^\gamma, \) \( B_{0\beta}^i = B_{\alpha\beta} v^\alpha. \) The quantities \( M_{\beta\gamma} \) and \( H_\beta \) are called the second fundamental v-tensor and normal curvature vector respectively ([6]). The second fundamental h-tensor \( H_{\beta\gamma} \) is defined as ([6])

\[ H_{\beta\gamma} = N_i (B_{\beta\gamma}^i + \Gamma^*_{jk} B_{\beta}^j B_{\gamma}^k) + M_{\beta} H_{\gamma}, \quad (26) \]

where

\[ M_{\beta} = N_i C_{jk}^i B_{\beta}^j N^k. \quad (27) \]

The relative h and v-covariant derivatives of projection factor \( B_{\alpha}^i \) with respect to \( ICT^* \) are given by

\[ B_{\alpha \beta}^i = H_{\alpha\beta} N^i, \quad B_{\alpha \beta}^i N^i = M_{\alpha\beta} N^i. \quad (28) \]

The equation (26) shows that \( H_{\beta\gamma} \) is generally not symmetric and

\[ H_{\beta\gamma} - H_{\gamma\beta} = M_{\beta} H_{\gamma} - M_{\gamma} H_{\beta}. \quad (29) \]

The above equation yield

\[ H_{0\gamma} = H_{\gamma}, \quad H_{\gamma 0} = H_{\gamma} + M_{\gamma} H_{0}. \quad (30) \]

We use following lemmas which are due to Matsumoto [6] as follows:

**Lemma 3.1.** The normal curvature \( H_0 = H_\beta v^\beta \) vanishes if and only if the normal curvature vector \( H_\beta \) vanishes.

**Lemma 3.2.** A hypersurface \( F^{n-1} \) is a hyperplane of the 1st kind if and only if \( H_\alpha = 0. \)

**Lemma 3.3.** A hypersurface \( F^{n-1} \) is a hyperplane of the 2nd kind with respect to the connection \( CT^* \) if and only if \( H_\alpha = 0 \) and \( H_{\alpha\beta} = 0. \)

**Lemma 3.4.** A hypersurface \( F^{n-1} \) is a hyperplane of the 3rd kind with respect to the connection \( CT^* \) if and only if \( H_\alpha = 0 \) and \( H_{\alpha\beta} = M_{\alpha\beta} = 0. \)
4 Hypersurface $F^{n-1}(c)$ of the special Finsler space

Let us consider special Finsler metric $L = \alpha + \beta + \beta^2 + \beta^3$ with a gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$ and a hypersurface $F^{n-1}(c)$ given by the equation $b(x) = c$ (constant) ([10]).

From parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get $\partial_i b(x(u)) = 0 = b_i B_\alpha^i$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$ we have

$$b_i B_\alpha^i = 0 \text{ and } b_i y^j = 0.$$  \hfill (31)

The induced metric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, \quad a_{\alpha\beta} = a_{ij} B_\alpha^i B_\beta^j$$  \hfill (32)

which is the Riemannian metric.

At a point of $F^{n-1}(c)$, from (5), (7) and (9), we have

$$p = 1, \quad q_0 = 2, \quad q_1 = 0, \quad q_2 = -\alpha^{-2}, \quad p_0 = 3, \quad p_1 = \alpha^{-1}, \quad p_2 = 0$$

$$\zeta = 1 + 2b^2, \quad S_0 = 2/(1 + 2b^2), \quad S_1 = 1/\alpha(1 + 2b^2), \quad S_2 = -b^2/\alpha^2(1 + 2b^2).$$  \hfill (33)

Therefore, from (8) we get

$$g^{ij} = a^{ij} - \frac{2}{1 + 2b^2} b^i b^j - \frac{1}{\alpha(1 + 2b^2)} (b^i y^j + b^j y^i) + \frac{b^2}{\alpha^2(1 + 2b^2)} y^i y^j.$$  \hfill (34)

Thus along $F^{n-1}(c)$, (34) and (31) lead to $g^{ij} b_i b_j = \frac{b^2}{1 + 2b^2}$.

Therefore, we get

$$b_i(x(u)) = \sqrt{\frac{b^2}{1 + 2b^2}} N_i, \quad b^2 = a^{ij} b_i b_j.$$  \hfill (35)

where $b$ is the length of the vector $b^i$.

Again from (34) and (35) we get

$$b^i = a^{ij} b_j = \sqrt{b^2(1 + 2b^2)} N_i + b^2 \alpha^{-1} y^i.$$  \hfill (36)

Thus we have

**Theorem 4.1.** In the special Finsler hypersurface $F^{n-1}(c)$, the induced Riemannian metric is given by (32) and the scalar function $b(x)$ is given by (35) and (36).
The angular metric tensor and metric tensor of $F^n$ are given by

$$h_{ij} = a_{ij} + 2b_ib_j - \frac{Y_iY_j}{\alpha^2},$$  \hspace{1cm} (37)

$$g_{ij} = a_{ij} + 3b_ib_j + \frac{1}{\alpha}(b_iY_j + b_jY_i).$$  \hspace{1cm} (38)

From (31), (37) and (18) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$, then along $F^{n-1}(c)$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$.

From (7), we get

$$\frac{\partial p_0}{\partial \beta} = 60\beta^3 + 12\alpha^3 + 36\alpha^2\beta + 40\alpha\beta^2.$$  \hspace{1cm} (39)

Thus along $F^{n-1}(c)$, $\frac{\partial p_0}{\partial \beta} = \frac{12}{\alpha}$ and therefore (11) gives $\gamma_1 = \frac{6}{\alpha}$, $m_i = b_i$.

Then the hv-torsion tensor becomes

$$C_{ijk} = \frac{1}{2\alpha}(h_{ij}b_k + h_{jk}b_i + h_{ki}b_j) + \frac{6}{\alpha}b_ib_jb_k$$  \hspace{1cm} (39)

in a special Finsler hypersurface $F^{n-1}(c)$.

Therefore, (18), (24), (27), (31) and (39) give

$$M_{\alpha\beta} = \frac{1}{2\alpha} \sqrt{\frac{b^2}{1+2b^2}}h_{\alpha\beta} \quad \text{and} \quad M_\alpha = 0.$$  \hspace{1cm} (40)

From (29) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

**Theorem 4.2.** The second fundamental v-tensor of special Finsler hypersurface $F^{n-1}(c)$ is given by (40) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.

Next from (31), we get $b_{i|\beta}B^i_\alpha + b_iB^i_{\alpha|\beta} = 0$. Therefore, from (28) and using $b_{i|\beta} = b_{i|j}B^j_\alpha + b_{i|j}N^jH_\beta$, we get

$$b_{i|j}B^i_\alpha B^j_\beta + b_{i|j}B^i_\alpha N^jH_\beta + b_iH_{\alpha\beta}N^i = 0.$$  \hspace{1cm} (41)

Since $b_{i|j} = -b_C^{ij}$, we get $b_{i|j}B^i_\alpha N^j = 0$.

Thus (41) gives

$$\sqrt{\frac{b^2}{1+2b^2}}H_{\alpha\beta} + b_{i|j}B^i_\alpha B^j_\beta = 0.$$  \hspace{1cm} (42)

It is noted that $b_{ij}$ is symmetric. Furthermore, contracting (42) with $v^\beta$ and then with $v^\alpha$ and using (15), (30) and (40), we get

$$\sqrt{\frac{b^2}{1+2b^2}}H_\alpha + b_{i|j}B^i_\alpha y^j = 0,$$  \hspace{1cm} (43)

$$\sqrt{\frac{b^2}{1+2b^2}}H_0 + b_{i|j}y^iy^j = 0.$$  \hspace{1cm} (44)
In view of Lemmas (3.1) and (3.2), the hypersurface \( F^{n-1}(c) \) is hyperplane of the first kind if and only if \( H_0 = 0 \). Thus from (44) it follows that \( F^{n-1}(c) \) is a hyperplane of the first kind if and only if \( b_{ij} y^i y^j = 0 \). Here \( b_{ij} \) being the covariant derivative with respect to \( CT \) of \( F^n \) depends on \( y^i \). On the other hand \( \nabla_j b_i = b_{ij} \) is the covariant derivative with respect to the Riemannian connection \( \left\{ \begin{array}{c} \iota \\ jk \end{array} \right\} \) constructed from \( a_{ij}(x) \), therefore \( b_{ij} \) does not depend on \( y^i \). We shall consider the difference \( b_{ij} - b_{ij} \) in the following. The difference tensor \( D_{jk}^i = \Gamma_{jk}^{\iota i} - \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \) is given by (13). Since \( b_i \) is a gradient vector, from (12) we have \( E_{ij} = b_{ij}, F_{ij} = 0 \) and \( F_{j}^{i} = 0 \). Thus (13) reduces to

\[
D_{jk}^i = B_{jk}^i - B_{0k}^i - B_{0i}^j + b_{0m} g^{im} b_{jk} - C_{jk}^{im} A^m_k - C_{km}^i A^m_j + C_{jk}^m A^m_s g^{is} + \lambda^i (C_{jk}^{im} C^m_s + C_{km}^i C^m_s - C_{jk}^m C^m_s m). \tag{45}
\]

In view of (33) and (34), the relations in (14) become to

\[
B_i = 3 b_i + \alpha^{-1} Y_i, \quad B^i = \frac{2 b^i}{1 + 2 b^2} + \frac{y^i}{\alpha (1 + 2 b^2)},
\]

\[
B_j^i = \frac{1}{2 \alpha} (\delta_j^i - \alpha^{-2} y^i Y_j) + \frac{5}{\alpha (1 + 2 b^2)} b_j b_i - \frac{1 + 12 b^2}{2 \alpha^2 (1 + 2 b^2)} y^i b_j, \tag{46}
\]

\[
B_{ij} = \frac{1}{2 \alpha} (a_{ij} - \alpha^{-2} Y_i Y_j + 12 b_i b_j),
\]

\[
A^m_k = B^m_k b_{00} + B^m b_{k0}, \quad \lambda^m = B^m b_{00}.
\]

By virtue of (46) we have \( B_i^0 = 0, B_{i0} = 0 \) which leads \( A^m_0 = B^m b_{00} \). Therefore we have

\[
D_{j0}^i = B_{j0}^i + B_{0j}^i - B^m C_{jm}^i b_{00}, \tag{47}
\]

\[
D_{00}^i = B_{00}^i = \left[ \frac{2 b^i}{1 + 2 b^2} + \frac{y^i}{\alpha (1 + 2 b^2)} \right] b_{00}. \tag{48}
\]

Thus from the relation (31), we get

\[
b_{ij} D_{j0}^i = \frac{2 b^2}{1 + 2 b^2} b_{j0} + \frac{1 + 12 b^2}{2 \alpha (1 + 2 b^2)} b_j b_{00} - \frac{2 b b^m b_{jm}^i C_{jm}^i b_{00}}{1 + 2 b^2}. \tag{49}
\]

\[
b_{ij} D_{00}^i = \frac{2 b^2}{1 + 2 b^2} b_{00}. \tag{50}
\]

From (39) it follows that

\[
b^m b_{ij} C_{jm}^i B_{00}^i = b^2 M_0 = 0.
\]
Therefore, the relation $b_{ij} = b_{ij} - b_rD^r_{ij}$ and equations (49), (50) give

$$b_{ij}y^i y^j = b_{00} - b_rD^r_{00} = \frac{1}{1 + 2b^2}b_{00}. \tag{51}$$

Consequently, (43) and (44) may be written as

$$\sqrt{b^2}H_0 + \frac{1}{\sqrt{1 + 2b^2}}b_{i0}B^i_\alpha = 0. \tag{52}$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where $b_{ij}$ does not depend on $y^i$. Since $y^i$ is to satisfy (31), the condition is written as $b_{ij}y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i. \tag{53}$$

From (31) and (53) it follows that $b_{00} = 0$, $b_{ij}B^i_\alpha B^j_\beta = 0$, $b_{ij}B^i_\alpha y^j = 0$. Hence (51) gives $H_\alpha = 0$. Again from (53) and (46) we get $b_{i0}b^i = \frac{\alpha b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij}B^i_\alpha B^j_\beta = \frac{1}{2\alpha}h_{\alpha\beta}$. Thus (26), (34), (35),(36),(40) and (45) give

$$b_rD^r_{ij}B^i_\alpha B^j_\beta = - \frac{C_0 b^2}{4\alpha(1 + 2b^2)}h_{\alpha\beta}. \tag{54}$$

Therefore, eq. (42) reduces to

$$\sqrt{\frac{b^2}{1 + 2b^2}}H_{\alpha\beta} + \frac{C_0 b^2}{4\alpha(1 + 2b^2)}h_{\alpha\beta} = 0. \tag{55}$$

Hence the hypersurface $F^{n-1}(c)$ is umbilic.

**Theorem 4.3.** The necessary and sufficient condition for $F^{n-1}(c)$ to be a hyperplane of 1st kind is (53) and in this case the second fundamental tensor of $F^{n-1}(c)$ is proportional to its angular metric tensor.

In view of Lemma (3.3) $F^{n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (54), we get $c_0 = c_i(x)y^i = 0$. Therefore, there exist a function $e(x)$ such that $c_i(x) = e(x)b_i(x)$. Thus (53) gives

$$b_{ij} = eb_i b_j. \tag{56}$$

**Theorem 4.4.** The necessary and sufficient condition for $F^{n-1}(c)$ to be a hyperplane of 2nd kind is (55).

Finally from (40) and Lemma (3.4) it is clear that $F^{n-1}(c)$ is not a hyperplane of third kind.

**Theorem 4.5.** The hypersurface $F^{n-1}(c)$ is not a hyperplane of the 3rd kind.
References


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