Arens Regularity of Some Banach Algebras

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Abstract. Let $A$ be a Banach algebra with the second dual $A^{**}$. If $A$ has a bounded approximate identity (= $BAI$), then $A^{**}$ is unital if and only if $A^{**}$ has a weak* bounded approximate identity (= $W*BAI$). If $A$ is Arens regular and $A$ has a BAI, then $A^*$ factors on the both sides.

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1. Preliminaries and Introduction

Through of this paper, $A$ is a Banach algebra and $A^*$, $A^{**}$, respectively, are the first and second dual of $A$. We say that a bounded net $(e_\alpha)_{\alpha \in I}$ in $A$ is an approximate identity (= $BAI$) if, for each $a \in A$, $ae_\alpha \to a$ and $e_\alpha a \to a$. For $a \in A$ and $f \in A^*$, we denote by $f.a$ and $aof$ respectively, the functionals on $A^*$ defined by $\langle f.a, b \rangle = \langle f, ab \rangle = f(ab)$ and $\langle aof, b \rangle = \langle f, ba \rangle = f(ba)$. The Banach algebra $A$ is embedded in its second dual via the identification $\langle a, f \rangle = \langle f.a, a \rangle$ for every $a \in A$ and $f \in A^*$. We denote by $A^*A$ and $AA^*$, respectively, the sets $\{f.a : a \in A and f \in A^*\}$ and $\{aof : a \in A and f \in A^*\}$. Hence, these two sets are subsets of $A^*$. We say that the Banach algebra $A$ is unital if there exists an element $e \in A$ such that $ex = xe = x$ for each $x \in A$. Let $A$ be a Banach algebra with a $BAI$. If the equality $A^*A = A^*$, $(AA^* = A^*)$ holds, then we say that $A^*$ factors on the left (right). If both equalities $A^*A = AA^* = A^*$ hold, then we say that $A^*$ factors on both sides.

Arens [1] has shown that given any Banach algebra $A$, there exist two algebra multiplications on the second dual of $A$ which extend multiplication on $A$. In the following, we introduce both multiplication which are given in [11]. Let $a, b \in A$ and $f \in A^*$ and $F, G \in A^{**}$ then the first Arens multiplication is defined as

$$\langle f.a, b \rangle = \langle f, ab \rangle,$$
$$\langle F.f, a \rangle = \langle F, f.a \rangle,$$
$$\langle F.G, f \rangle = \langle F, G.f \rangle.$$
Clearly $F,f \in A^*$ and $F,G \in A^{**}$. We use the notions $(A^{**},.)$ for $A^{**}$ equipped with first Arens multiplication also we use $fa$ instead of $f.a$ for all $a \in A$ and $f \in A^*$.

The second Arens product is defined as follows
For $a,b \in A$, $f \in A^*$ and $F,G \in A^{**}$ the elements $aof$, $foF$ of $A^*$ and $FoG$ of $A^{**}$ are defined respectively by the equalities

$$\langle aof, b \rangle = \langle f, ba \rangle,$$

$$\langle foF, a \rangle = \langle F, aof \rangle,$$

$$\langle FoG, f \rangle = \langle G, foF \rangle.$$

An element $E$ of $A^{**}$ is said to be a mixed unit if $E$ is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, $E$ is a mixed unit if and only if, for each $F \in A^{**}$, $F.E = EoF = F$.

We say that $A^{**}$ is unital with respect to the first Arens product, if there exists an element $E \in A^{**}$ such that $F.E = E.F = F$ for all $F \in A^{**}$ , and $A^{**}$ is unital with respect to the second Arens product, if there exists an element $E \in A^{**}$ such that $FoE = EoF = F$ for all $F \in A^{**}$. By [3, p.146], an element $E$ of $A^{**}$ is mixed unit if and only if it is a weak* cluster point of some BAI $(e_\alpha)_{\alpha \in I}$ in $A$.

Suppose that $F,G \in A^{**}$ and $F,G$, $FoG$ are the first and second Arens Multiplications in $A^{**}$, respectively. Then the mapping $F \rightarrow F.G$, for $G$ fixed in $A^{**}$, is weak* – to – weak* continuous, but the mapping $F \rightarrow G.F$ for $G$ fixed in $A^{**}$ is not in general weak* – to – weak* continuous on $A^{**}$ unless $G \in A$.

As an example for the algebras $L^1(G)^{**}$ and $M(G)^{**}$ whenever $G$ is an infinite topological group the mapping $F \rightarrow G.F$, in general, is not weak* – to – weak* continuous on $L^1(G)^{**}$ and $M(G)^{**}$, see [10, 11]. Hence, the first topological center of $A^{**}$ with respect to first Arens product is defined as follows

$$Z_1 = \{ G \in A^{**} : F \rightarrow G.F \text{ is weak}^* – \text{to} – \text{weak}^* – \text{continuous on } A^{**} \}.$$ 

Also, for the second Arens multiplication in $A^{**}$ we know that $aof$, $foF \in A^*$ and $FoG \in A^{**}$. For $G$ fixed in $A^{**}$, the mapping $F \rightarrow GoF$ is weak* – to – weak* continuous on $A^{**}$, but the mapping $F \rightarrow FoG$ is not in general weak* – to – weak* continuous on $A^{**}$ unless $G \in A$. Whence, the second topological center of $A^{**}$ with respect to second Arens product is defined as follows

$$Z_2 = \{ G \in A^{**} : F \rightarrow FoG \text{ is weak}^* – \text{to} – \text{weak}^* – \text{continuous on } A^{**} \}.$$ 

It is clear that $A \subseteq Z_1 \cap Z_2$ and $Z_1$, $Z_2$ are closed subalgebras of $A^{**}$ endowed with the first second Arens multiplication, respectively.

If, for each $F,G \in A^{**}$, the equality $F.G = FoG$ holds, then the algebra $A$ is said to be Arens regular, see [1, 2]. In this case $Z_1 = Z_2 = A^{**}$.

The other extreme situation is that $Z_1 = A^*$ , in this case $A$ is called left strongly Arens irregular, see [9, 10, 14].

We recall that the topological center of $A^{**}$ is defined the set of all functionals
F \in A^{**} which satisfy F.G = FoG for all G \in A^{**}, see [11]. In other words, the topological centers of A^{**} with respect to the first and second Arens products can also be defined as following sets respectively

\[ Z_1 = \{ F \in A^{**} : F.G = FoG \forall G \in A^{**} \}, \]

\[ Z_2 = \{ F \in A^{**} : G.F = GoF \forall G \in A^{**} \}. \]

For a Banach algebra A the topological center of the algebra \((A^*A)^*\) is defined as follows, see [11].

\[ \tilde{Z} = \{ \mu \in (A^*A)^* : \lambda \rightarrow \lambda.\mu \text{ is weak}^* \text{ to weak}^* \text{ - continuous on } (A^*A)^* \}. \]

For all \( a \in A, f \in A^* \) and \( F \in A^{**} \) we have the following statements

i) \( f.a = foa \) and \( a.f = aof \).

ii) \( a.F = aoF \) and \( F.a = Foa \).

Take \( b \in A \), then

\[ \langle foa, b \rangle = \langle b, foa \rangle = \langle ab, f \rangle = f(ab). \]

Thus \( \langle f.a, b \rangle = f(ab) \), and we conclude that \( f.a = foa \).

Consequently we can write

\[ \langle a.F, f \rangle = \langle a, F.f \rangle = \langle F.f, a \rangle = \langle F, f.a \rangle = \langle F, foa \rangle = \langle aoF, f \rangle. \]

Therefore \( a.F = aoF \).

A functional \( f \) in \( A^* \) is said to be \( \text{wap} \) (weakly almost periodic) on \( A \) if the mapping \( a \rightarrow f.a \) from \( A \) into \( A^* \) is weakly compact. Pym in [15] showed that this definition to the equivalent following condition

For any two nets \((a_\alpha)\alpha\) and \((b_\beta)\beta\) in \( \{ a \in A : \| a \| \leq 1 \} \), we have

\[ \lim_{\alpha} \lim_{\beta} \langle f, a_\alpha b_\beta \rangle = \lim_{\beta} \lim_{\alpha} \langle f, a_\alpha b_\beta \rangle, \]

whenever both iterated limits exist. The collection of all \( \text{wap} \) functionals on \( A \) is denoted by \( \text{wap}(A) \). So \( f \in \text{wap}(A) \) if and only if \( \langle F.G, f \rangle = \langle F, GoF \rangle \) for every \( F, G \in A^{**} \), see [15, 19].

In this paper, the notations \( W^{SC} \) is used for weakly sequentially complete Banach algebra \( A \), that is, \( A \) is said to be weakly sequentially complete, if every weakly Cauchy sequence in \( A \) has a weak limit.

2. Main results

We say that \( A^{**} \) has a \( \text{weak}^* \text{ bounded left approximate identity} (= \text{W}^*\text{BLAI}) \) with respect to the first Arens product, if there is a bounded net as \((e_\alpha)\alpha \subseteq A\) such that for all \( F \in A^{**} \) and \( f \in A^* \), we have \( \langle e_\alpha.F, f \rangle \rightarrow \langle F, f \rangle \). The definition of \( \text{W}^*\text{RBAI} \) is similar and if \( A^{**} \) has both \( \text{W}^*\text{LBAI} \) and \( \text{W}^*\text{RBAI} \), then we say that \( A^{**} \) has \( \text{W}^*\text{BAI} \).
Theorem 2-1. Let $A$ has a BAI $(\epsilon_\alpha)$. Then

1. $A^*$ factors on the left if and only if $A^{**}$ has a $\text{W}^*\text{LBAI}$.
2. $A^*$ factors on the right if and only if $A^{**}$ has a $\text{W}^*\text{RBAI}$.
3. If $A^*$ factors on the left and $E \in A^{**}$ such that $\epsilon_\alpha \xrightarrow{w^*} E$, then $E$ is unit element of $A^{**}$. Moreover, if $A^*$ factors on the right and $E \in A^{**}$ such that $\epsilon_\alpha \xrightarrow{w^*} E$, then $E$ is unit element of $(A^{**}, o)$.

Proof. 1. Let $A^*$ factors on the left, and $F \in A^{**}$, $f \in A^*$. Then, by [11, Lemma 2.1], we have $\langle \epsilon_\alpha F, f \rangle = \langle F, \epsilon_\alpha f \rangle = \langle F, f \epsilon_\alpha \rangle \rightarrow \langle F, f \rangle$.

Consequently, $\epsilon_\alpha F \xrightarrow{w^*} F$.

Conversely, let $\epsilon_\alpha F \xrightarrow{w^*} F$ for all $F \in A^{**}$. Then, for every $f \in A^*$, we have $\langle \epsilon_\alpha F, f \rangle \rightarrow \langle F, f \rangle$. Since $\langle \epsilon_\alpha F, f \rangle = \langle F, \epsilon_\alpha f \rangle$, we have $\langle F, f \epsilon_\alpha \rangle \rightarrow \langle F, f \rangle$. Therefore by [11, Lemma 2.1], we are done.

2. Proof is the same as (2).

3. Let $\epsilon_\alpha \xrightarrow{w^*} E$. Then, for all $F \in A^{**}$, by using [11, Lemma 2.1], we have

$$\langle E F, f \rangle = \langle E, F f \rangle = \lim_{\alpha} \langle \epsilon_\alpha F, f \rangle = \lim_{\alpha} \langle F, \epsilon_\alpha f \rangle \rightarrow \langle F, f \rangle.$$ 

Hence, we conclude that $E F = F$.

Now let $f \in A^*$ and $a \in A$. Therefore

$$\langle E f, a \rangle = \langle E, f a \rangle = \lim_{\alpha} \langle \epsilon_\alpha F, a \rangle = \lim_{\alpha} \langle f, a \epsilon_\alpha \rangle = \langle f, a \rangle.$$ 

It follows that $\langle F E, f \rangle = \langle F, E f \rangle = \langle F, f \rangle$. Hence $F E = F$ The next assertion is similar.

By using preceding theorem and by [11, Proposition 2.2.a], for a Banach algebra $A$ which has BAI, we conclude that $A^{**}$ has $\text{W}^*\text{LBAI}$ if and only if $\{ F \in A^{**} : A F \subseteq A \} \subseteq Z_1$. Moreover $A^{**}$ has $\text{W}^*\text{RBAI}$ if and only if $\{ F \in A^{**} : FoA \subseteq A \} \subseteq Z_2$. For $A = L^1(G)$ where $G$ is locally compact finite group, we have $\{ F \in A^{**} : A F \subseteq A \} \subseteq Z_1 = L^1(G)^{**}$, and so $L^1(G)^{**}$ has $\text{W}^*\text{LBAI}$. But when $G$ in locally compact infinite group the preceding theorem is not true. Hence $L^1(G)^{**}$ has not $\text{W}^*\text{LBAI}$.

Corollary 2-2. Let $A$ has BAI. Then the following statements hold.

1. $(A^{**}, .)$ is unital if and only if $A^{**}$ has $\text{W}^*\text{LBAI}$.
2. $(A^{**}, o)$ is unital if and only if $A^{**}$ has $\text{W}^*\text{RBAI}$.
3. $A^{**}$ and $(A^{**}, o)$ are unital if and only if $A^{**}$ has $\text{W}^*\text{BAI}$. 

By using preceding theorem and by [11, Proposition 2.2.a], for a Banach algebra $A$ which has BAI, we conclude that $A^{**}$ has $\text{W}^*\text{LBAI}$ if and only if $\{ F \in A^{**} : A F \subseteq A \} \subseteq Z_1$. Moreover $A^{**}$ has $\text{W}^*\text{RBAI}$ if and only if $\{ F \in A^{**} : FoA \subseteq A \} \subseteq Z_2$. For $A = L^1(G)$ where $G$ is locally compact finite group, we have $\{ F \in A^{**} : A F \subseteq A \} \subseteq Z_1 = L^1(G)^{**}$, and so $L^1(G)^{**}$ has $\text{W}^*\text{LBAI}$. But when $G$ in locally compact infinite group the preceding theorem is not true. Hence $L^1(G)^{**}$ has not $\text{W}^*\text{LBAI}$.
4. Suppose that $A$ is a WSC with sequentially BAI. Then $A^{**}$ has $W^*LBAI$ if and only if $A^{**}$ has $W^*RBAI$, moreover $(A^{**},.)$ is unital if and only if $(A^{**},o)$ is unital.

Proof. By using Theorem 2-1 and [11, Proposition 2.2], the proofs of (1), (2) are clear. The proof of (4), follows from [11, Proposition 2.6] and Theorem 2-1.

As another application of Theorem 2-1, let $\Omega$ be spectrum of $L^\infty([0,1])$. Since $L^\infty([0,1])$ not factors on the left and right, then by Theorem 1, $L^1([0,1])^{**}$ has not $W^*LBAI$ and $W^*RBAI$, and so $M(\Omega) = L^1([0,1])$ has $W^*LBAI$ and $W^*RBAI$. Consequently by Corollary 2-2, $M(\Omega)$ is not unital which follows that $\Omega$ is not a group. For more details see [9, 16].

Corollary 2-3. Let $A$ be Arens regular with a BAI. Assume also that $A$ is a two sided ideal in $A^{**}$, then $A^{**}$ has a W*BAI and so is unital.

Proof. By use Theorem 1 and [17, Theorem 3.2], proof hold.

Theorem 2-4. Suppose $f \in wap(A)$ and $G \in A^{**}$ such that $a_\alpha \overset{w^*}{\to} G$ where $(a_\alpha)_\alpha \in A$. Then $f.a_\alpha \overset{w}{\to} f.G$.

Proof. First we show that $f.a_\alpha \overset{w^*}{\to} f.G$. Let $b \in A$, so

$$\langle f.a_\alpha, b \rangle = \langle a_\alpha, b.f \rangle \to \langle G, b.f \rangle = \langle f.G, b \rangle.$$ 

Thus $f.a_\alpha \overset{w^*}{\to} f.G$.

Since $f \in wap(A)$, $\langle G.F, f \rangle = \langle GoF, f \rangle$ for all $F, G \in A^{**}$. We have

$$\langle f.G, F \rangle = \langle G.F, f \rangle = \langle GoF, f \rangle = \langle F, f.oG \rangle = \langle f.oG, F \rangle,$$

it follows that $f.G = f.oG \in A^*$. For every $F \in A^{**}$, there is $(b_{\beta})_{\beta} \subseteq A$ such that $b_{\beta} \overset{w^*}{\to} F$, and so we have

$$\lim_{\alpha} \langle F, f.a_\alpha \rangle = \lim_{\alpha} \lim_{\beta} \langle b_{\beta}, f.a_\alpha \rangle = \lim_{\alpha} \lim_{\beta} \langle f, a_\alpha b_{\beta} \rangle,$$

$$= \lim_{\beta} \lim_{\alpha} \langle f, a_\alpha b_{\beta} \rangle = \lim_{\beta} \langle f.G, b_{\beta} \rangle = \lim_{\beta} \langle b_{\beta}, f.G \rangle,$$

$$= \langle F, f.G \rangle.$$

Therefore $f.a_\alpha \overset{w}{\to} f.G$. 

Corollary 2-5. Suppose $A$ has a BAI $(e_\alpha)_\alpha$ and $f \in wap(A)$. Then $f.e_\alpha \overset{w}{\to} f$. 

Proof. By [5, p.146], there is a mixed unit \( E \in A^{**} \) such that \( e_\alpha \overset{w^*}{\to} E \). Since \( f \in \text{wap}(A) \), \( \langle E.F, f \rangle = \langle EoF, f \rangle = \langle F, f \rangle \) for all \( F \in A^{**} \). So, by Theorem 2-4, we have \( f.e_\alpha \overset{w}{\to} f.E = f \).

Corollary 2-6. Suppose \( A \) is Arens regular and \( A \) has a BAI. Then we have the following assertions.
1. \( A^* \) factors on both sides.
2. \( A^{**} \) is unital, and if \( A \) is WSC then \( A \) is unital.

Proof. 1. Since \( A \) is Arens regular, \( \text{wap}(A) = A^* \) hence by Corollary 2-5, we have \( f.e_\alpha \overset{w}{\to} f \) for all \( f \in A^* \). Then, it follows that \( A^* \) factors on the left by [11, Lemma 2.1]. Since \( A \) is Arens regular, by [11, Proposition 2.10] \( A^* \) factors on the right and so \( A^* \) factors on both sides.

2. By [11, Proposition 2.2], since \( A^* \) factors on both sides, \((A^{**},.)\) and \((A^{**},o)\) are unital. The second claim hold by using [2, Proposition 2.6] and part (1).

Since \( L^1(G)^* \) does not factors on the left and right whenever \( G \) is infinite non-discrete group, by Corollary 2-6, we conclude that \( L^1(G)^{**} \) is not Arens regular.

Theorem 2-7. For a Banach algebra \( A \) we have the following assertions.
1. Assume that \( f \in A^* \) and \( T_f : A^{**} \to A^* \) be the adjoint of \( T_f \). For \( F \in A^{**} \) and \( a \in A \), we have \( \langle T_f F, a \rangle = \langle F, T_f a \rangle \).

\[ \langle G, T_f^* F_\alpha \rangle = \lim_\beta \langle T_f^* F_\alpha, a_\beta \rangle = \lim_\beta \langle F_\alpha, T a_\beta \rangle = \lim_\beta \langle F_\alpha, f.a_\beta \rangle \]
\[ = \lim_\beta \langle F_\alpha.f, a_\beta \rangle = \langle G, F_\alpha.f \rangle, \]

hence \( \lim_\alpha \langle G, T_f^* F_\alpha \rangle = \lim_\alpha \langle G, F_\alpha.f \rangle = \lim_\alpha \langle G.F_\alpha, f \rangle = \langle G.F, f \rangle \)

Proof. 1. Let \( f \in \text{wap}(A) \) and \( T_f^* : A^{**} \to A^* \) be the adjoint of \( T_f \). Thus for every \( F \in A^{**} \) and \( a \in A \), we have \( \langle T_f^* F, a \rangle = \langle F, T_f a \rangle \).

Suppose \( (F_\alpha)_\alpha \subseteq A^{**} \) such that \( F_\alpha \overset{w^*}{\to} F \). We show that \( T_f^* F_\alpha \overset{w}{\to} T_f^* F \).

Let \( G \in A^{**} \) and \( (a_\beta)_\beta \subseteq A \) such that \( a_\beta \overset{w^*}{\to} G \). Since \( f \in \text{wap}(A) \), we have \( \langle G.F_\alpha, f \rangle \to \langle G.F, f \rangle \). Hence for fixed \( \alpha \) we have the following relations

\[ \langle G, T_f^* F_\alpha \rangle = \lim_\beta \langle T_f^* F_\alpha, a_\beta \rangle = \lim_\beta \langle F_\alpha, T a_\beta \rangle = \lim_\beta \langle F_\alpha, f.a_\beta \rangle \]
\[ = \lim_\beta \langle F_\alpha.f, a_\beta \rangle = \langle G, F_\alpha.f \rangle, \]

hence \( \lim_\alpha \langle G, T_f^* F_\alpha \rangle = \lim_\alpha \langle G, F_\alpha.f \rangle = \lim_\alpha \langle G.F_\alpha, f \rangle = \langle G.F, f \rangle \).
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\[ = \langle G, F.f \rangle = \langle G, T_f^*F \rangle. \]

We conclude that \( T_f^*F_{\alpha} \xrightarrow{w^*} T_f^*F \), thus \( T_f^* \) is \( \text{weak }^* - \text{weak} \) continuous.

Conversely, let \( T_{a}^* \) be \( \text{weak }^* - \text{weak} \) continuous and \( (F_{\alpha})_{\alpha} \subseteq A^{**} \) such that \( F_{\alpha} \xrightarrow{w^*} F \). Then for every \( G \in A^{**} \), we have

\[ \langle G.F_{\alpha}, f \rangle = \langle G, T_{a}^*F_{\alpha} \rangle \rightarrow \langle G, T_{a}^*F \rangle = \langle G, F, f \rangle. \]

It follow from this that \( f \in \text{wap}(A) \).

2. Let \( \tilde{Z} = (A^*A)^* \) and \( F \in A^{**} \). We define \( \mu \in (A^*A)^* \) to be the restriction of \( F \) to \( (A^*A)^* \). Since \( \tilde{Z} = (A^*A)^* \), \( \mu \in \tilde{Z} \), and so for all \( a \in A \), \( a\mu = aF \). By [11, Proposition 3.2], since \( AZ = AZ_1 \) and \( AZ_1 \subseteq Z_1 \), we have \( aF \in Z_1 \). Therefore, for every \( G \in A^{**} \) and \( f \in A^* \) we have the following relations

\[ \langle F.G, f.a \rangle = \langle aF.G, f \rangle = \langle aFoG, f \rangle = \langle FoG, f.a \rangle. \]

Consequently \( f.a \in \text{wap}(A) \).

Theorem 2-7 shows that a Banach algebra \( A \) is Arens regular if and only if \( T_f^* \) is \( \text{weak }^* - \text{weak} \) continuous for every \( f \in A^* \).

References


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