Int. J. Contemp. Math. Sciences, Vol. 4, 2009, no. 11, 505 - 518

Common Fixed Point Theorem in Intuitionistic Fuzzy Metric Space Using General Contractive Condition of Integral Type

S. Muralisankar and G. Kalpana

Department of Mathematics, Anna University Chennai
Chennai-600 025, India
muralipsna@yahoo.co.in

Abstract

The aim of this paper is to obtain a common fixed point theorem in an intuitionistic fuzzy metric space for pointwise R-weakly commuting mappings using contractive condition of integral type and to establish a situation in which a collection of maps has a fixed point which is a point of discontinuity.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Fuzzy set, Intuitionistic fuzzy set, Intuitionistic fuzzy metric space, pointwise R-weakly commuting, reciprocally continuous, non-compatible, Integral type

1 Introduction

Atanassov [3] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [21] and later there has been much progress in the study of intuitionistic fuzzy sets by many authors [4, 7]. In 2004, Park [16] introduced a notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [12]. Fixed point theory has important applications in diverse disciplines of mathematics, statistics, engineering and economics in dealing with problems arising in : Approximation theory, potential theory, game theory, mathematical economics, etc. Several authors [9, 10, 12, 13, 18] proved some fixed point theorems for various generalizations of contraction mappings in probabilistic and fuzzy metric space. Branciari [6] obtained a fixed point theorem for a single mapping satisfying an analogue of

Banach's contraction principle for an integral type inequality. Sedghi.at.el [19] established a common fixed point theorem for weakly compatible mappings in intuitionistic fuzzy metric space satisfying a contractive condition of integral type.
In this paper, we prove a common fixed point theorem in an intuitionistic fuzzy metric space for pointwise R-weakly commuting mappings using contractive condition of integral type and to establish a situation in which a collection of maps has a fixed point which is a point of discontinuity.

2 Preliminaries

Definition 2.1. *[21] Let X be any set. A fuzzy set A in X is a function with domain X and values in* $[0, 1]$.

Definition 2.2. *[3] Let a set E be fixed. An intuitionistic fuzzy set (IFS) A of E is an object having the form,*

$$A = \{< x, \mu_A(x), V_A(x) > /x \in E\}$$

where the function $\mu_A : E \to [0, 1]$, $V_A : E \to [0, 1]$ define respectively, the degree of membership and degree of non-membership of the element $x \in E$ to the set A, which is a subset of E, and for every $x \in E$, $0 \leq \mu_A(x)+V_A(x) \leq 1$.

Definition 2.3. *[18] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:*

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a * 1 = a$ for all $a \in [0, 1]$;

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Definition 2.4. *[18] A binary operation $\diamondsuit : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-conorm if it satisfies the following conditions:*

(a) $\diamondsuit$ is commutative and associative;

(b) $\diamondsuit$ is continuous;

(c) $a\diamondsuit 0 = a$ for all $a \in [0, 1]$;

(d) $a\diamondsuit b \leq c\diamondsuit d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Definition 2.5. *[1] A 5-tuple $(X, M, N, *, \diamondsuit)$ is said to be an intuitionistic fuzzy metric space (shortly IFM-Space) if X is an arbitrary set, $*$ is a continuous*
t-norm, $\diamondsuit$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$;

(IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$;

(IFM-2) $M(x, y, 0) = 0$;

(IFM-3) $M(x, y, t) = 1$ *if and only if* $x = y$;

(IFM-4) $M(x, y, t) = M(y, x, t)$;

(IFM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(IFM-6) $M(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous;*

(IFM-7) $lim_{t\to\infty} M(x, y, t) = 1$;

(IFM-8) $N(x, y, 0) = 1$;

(IFM-9) $N(x, y, t) = 0$ *if and only if* $x = y$;

(IFM-10) $N(x, y, t) = N(y, x, t)$;

(IFM-11) $N(x, y, t) \diamondsuit N(y, z, s) \geq N(x, z, t + s)$;

(IFM-12) $N(x, y, .) : [0, \infty) \to [0, 1]$ *is right continuous;*

(IFM-13) $lim_{t\to\infty} N(x, y, t) = 0$;

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and degree of non-nearness between x and y with respect to t, respectively.

Remark 2.6. *Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space if X of the form $(X, M, 1 - M, *, \diamondsuit)$ such that t- norm $*$ and t-conorm $\diamondsuit$ are associated, that is, $x \diamondsuit y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in X$. But the converse is not true.*

Example 2.7. *[16] Let (X, d) be a metric space. Denote $a * b = ab$ and $a \diamondsuit b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows;*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then (M_d, N_d) is an intuitionistic fuzzy metric on X. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Remark 2.8. *Note the above example holds even with the t-norm* $a * b = \min\{a, b\}$ *and the t-conorm* $a \diamondsuit b = \max\{a, b\}$ *and hence* (M_d, N_d) *is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm.*

Example 2.9. *Let* $X = N$. *Define* $a * b = \ max\{0, a + b - 1\}$ *and* $a \diamondsuit b = a + b - ab$ *for all* $a, b \in [0, 1]$ *and let* M *and* N *be fuzzy sets on* $X^2 \times (0, \infty)$ *defined as follows;*

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y, \\ \\ \frac{y}{x} & \text{if } y \leq x, \end{cases}$$

$$N(x, y, t) = \begin{cases} \frac{y-x}{y} & \text{if } x \leq y, \\ \\ \frac{x-y}{x} & \text{if } y \leq x, \end{cases}$$

for all $x, y, z \in X$ *and* $t > 0$. *Then* $(X, M, N, *, \diamondsuit)$ *is an intuitionistic fuzzy metric space.*

Remark 2.10. *Note that, in the above example, t-norm* $*$ *and t-conorm* $\diamondsuit$ *are not associated. And there exists no metricd on* X *satisfying*

$$M(x, y, t) = \frac{t}{t + d(x, y)}, N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

where $M(x, y, t)$ *and* $N(x, y, t)$ *are as defined in above example. Also note the above function* (M, N) *is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as* $a * b = \min\{a, b\}$ *and* $a \diamondsuit b = \max\{a, b\}$.

Definition 2.11. *[1] Let* $(X, M, N, *, \diamondsuit)$ *be an intuitionistic fuzzy metric space.*

(a) A sequence $\{x_n\}$ *in* X *is called cauchy sequence if for each* $t > 0$ *and* $P > 0$, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ *and* $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$.

(b) A sequence $\{x_n\}$ *in* X *is convergent to* $x \in X$ *if* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *and* $\lim_{n\to\infty} N(x_n, x, t) = 0$ *for each* $t > 0$.

(c) An intuitionistic fuzzy metric space is said to be complete if every Cauchy sequence is convergent.

Lemma 2.12. *[16] In an intuitionistic fuzzy metric spaceX,* $M(x, y, .)$ *is non-decreasing and* $N(x, y, .)$ *is non-increasing for all* $x, y \in X$.

Lemma 2.13. *[20] Let* $(X, M, N, *, \diamondsuit)$ *be an intuitionistic fuzzy metric space. If there exists a constant* $k \in (0, 1)$ *such that*

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t),$$

$$N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$$

$\forall t > 0$ *and* $n = 1, 2, ..$ *then* $\{y_n\}$ *is a cauchy sequence in* X.

Lemma 2.14. *[20] Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. If there exists a constant $k \in (0,1)$ such that*

$$M(x, y, kt) \geq M(x, y, t), N(x, y, kt) \leq N(x, y, t),$$

for $x, y \in X$. Then $x = y$.

Definition 2.15. *[14] Let (X, d) be a metric space. Two self mappings f and g of X are said to be R-weakly commuting if there exists a positive real number $R > 0$ such that*

$$d(fg(x), gf(x)) \leq Rd(f(x), g(x))$$

for all $x \in X$.

Definition 2.16. *Let $(X, M, N, *, \diamondsuit)$ be an intuitionistic fuzzy metric space. Two self mappings f and g of X are said to be pointwise R-weakly commuting on X if given $x \in X$ there exists a positive real number $R > 0$ such that*

$$M(fg(x), gf(x), t) \geq M(f(x), g(x), t/R)$$

$$N(fg(x), gf(x), t) \leq N(f(x), g(x), t/R)$$

and $t > 0$.

Definition 2.17. *Let A and S be mappings from an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ into itself. Then the mappings are said to be compatible if*

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1,$$

$$\lim_{n\to\infty} N(ASx_n, SAx_n, t) = 0,$$

for every $t > 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Definition 2.18. *Let A and S be mappings from an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ into itself. Then the mappings are said to be non-compatible if whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$. But

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) \neq 1$$

or non-existent,

$$\lim_{n\to\infty} N(ASx_n, SAx_n, t) \neq 0$$

or non-existent.

Definition 2.19. *Let A and S be mappings from an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ into itself. Then the mappings are said to be reciprocally continuous if*

$$\lim_{n\to\infty} ASx_n = Az, \quad \text{and} \quad \lim_{n\to\infty} SAx_n = Sz,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Remark 2.20. *If A and S are both continuous then they are obviously reciprocally continuous. But the converse need not be true.*

3 Main Results

Theorem 3.1. *Let (A, S) and (B, T) be a pointwise R-weakly commuting pairs of selfmappings of a complete intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ with continuous t-norm $*$ and continuous t-corm $\diamondsuit$ defined by $t * t \geq t$ and $(1-t)\diamondsuit(1-t) \leq (1-t)$ for all $t \in [0, 1]$ such that,*

(i) $AX \subset TX, BX \subset SX$

(ii) there exists a constant $k \in (0, 1)$ such that

$$\int_0^{M(Ax,By,kt)} \varphi(t)dt \geq \Big(\int_0^{m(x,y,t)} \varphi(t)dt\Big), \tag{3.1}$$

$$\int_0^{N(Ax,By,kt)} \varphi(t)dt \leq \Big(\int_0^{n(x,y,t)} \varphi(t)dt\Big), \tag{3.2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable, nonnegative, and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad for \quad each \quad \epsilon > 0,$$

where

$$m(x, y, t) = \min\{M(Ty, By, t), M(Sx, Ax, t), M(Sx, By, \alpha t), M(Ty, Ax, (2-\alpha)t), M(Ty, Sx, t)\}$$

$$n(x,y,t) = \max\{N(Ty,By,t), N(Sx,Ax,t), N(Sx,By,\alpha t), N(Ty,Ax,(2-\alpha)t), N(Ty,Sx,t)\}$$

for all $x, y \in X$, $\alpha \in (0,2)$ and $t > 0$. Suppose that (A, S) or (B,T) is a compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. we construct a sequence $\{y_n\}$ in X such that for $n = 0, 1, 2...$

$$\begin{aligned} y_{2n} &= Ax_{2n} = Tx_{2n+1} \\ y_{2n+1} &= Bx_{2n+1} = Sx_{2n+2}. \end{aligned} \tag{3.3}$$

We show that $\{y_n\}$ is a Cauchy sequence. By (3.1) and (3.2), for all $t > 0$ and $\alpha = 1 - \beta$ with $\beta \in (0,1)$, we have

$$\begin{aligned} \int_0^{M(y_{2n+1},y_{2n+2},kt)} \varphi(t)dt &= \int_0^{M(Bx_{2n+1},Ax_{2n+2},kt)} \varphi(t)dt, \\ &= \int_0^{M(Ax_{2n+2},Bx_{2n+1},kt)} \varphi(t)dt, \\ &\geq \int_0^{m(x_{2n+2},x_{2n+1},t)} \varphi(t)dt, \end{aligned}$$

$$\begin{aligned} \int_0^{N(y_{2n+1},y_{2n+2},kt)} \varphi(t)dt &= \int_0^{N(Bx_{2n+1},Ax_{2n+2},kt)} \varphi(t)dt, \\ &= \int_0^{N(Ax_{2n+2},Bx_{2n+1},kt)} \varphi(t)dt, \\ &\leq \int_0^{n(x_{2n+2},x_{2n+1},t)} \varphi(t)dt. \end{aligned}$$

$$\begin{aligned} m(x_{2n+2},x_{2n+1},t) &= \min\{M(Tx_{2n+1},Bx_{2n+1},t), M(Ax_{2n+2},Sx_{2n+2},t), M(Sx_{2n+2},Bx_{2n+1},\alpha t), \\ &\quad M(Tx_{2n+1},Ax_{2n+2},(2-\alpha)t), M(Tx_{2n+1},Sx_{2n+2},t)\} \\ &= \min\{M(y_{2n},y_{2n+1},t), M(y_{2n+1},y_{2n+2},t), M(y_{2n+1},y_{2n+1},\alpha t), \\ &\quad M(y_{2n},y_{2n+2},(1+\beta)t)), M(y_{2n},y_{2n+1},t)\} \\ &\geq \min\{M(y_{2n},y_{2n+1},t), M(y_{2n+1},y_{2n+2},t), 1, \\ &\quad M(y_{2n},y_{2n+1},t), M(y_{2n+1},y_{2n+2},\beta t), M(y_{2n},y_{2n+1},t)\} \\ &\geq \min\{M(y_{2n},y_{2n+1},t), M(y_{2n+1},y_{2n+2},t), \\ &\quad M(y_{2n+1},y_{2n+2},\beta t)\} \end{aligned}$$

$$\begin{aligned} n(x_{2n+2},x_{2n+1},t) &= \max\{N(Tx_{2n+1},Bx_{2n+1},t), N(Ax_{2n+2},Sx_{2n+2},t), N(Sx_{2n+2},Bx_{2n+1},\alpha t), \\ &\quad N(Tx_{2n+1},Ax_{2n+2},(2-\alpha)t), N(Tx_{2n+1},Sx_{2n+2},t)\} \\ &= \max\{N(y_{2n},y_{2n+1},t), N(y_{2n+1},y_{2n+2},t), N(y_{2n+1},y_{2n+1},\alpha t), \\ &\quad N(y_{2n},y_{2n+2},(1+\beta)t)), N(y_{2n},y_{2n+1},t)\} \\ &\leq \max\{N(y_{2n},y_{2n+1},t), N(y_{2n+1},y_{2n+2},t), 1, \\ &\quad N(y_{2n},y_{2n+1},t), N(y_{2n+1},y_{2n+2},\beta t), N(y_{2n},y_{2n+1},t)\} \\ &\leq \max\{N(y_{2n},y_{2n+1},t), N(y_{2n+1},y_{2n+2},t), \\ &\quad N(y_{2n+1},y_{2n+2},\beta t)\} \end{aligned}$$

since t-norm $*$, t-conorm $\diamondsuit$, $M(x,y,.)$ and $N(x,y,.)$ is continuous. Letting $\beta \to 1$, we have

$$m(x_{2n+2}, x_{2n+1}, t) \geq \min\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t)\}$$

$$n(x_{2n+2}, x_{2n+1}, t) \leq \max\{N(y_{2n}, y_{2n+1}, t), N(y_{2n+1}, y_{2n+2}, t)\}$$

Therefore,

$$\int_0^{M(y_{2n+1},y_{2n+2},kt)} \varphi(t)dt \;\geq\; \int_0^{\min\{M(y_{2n},y_{2n+1},t),M(y_{2n+1},y_{2n+2},t)\}} \varphi(t)dt,$$

$$\int_0^{N(y_{2n+1},y_{2n+2},kt)} \varphi(t)dt \;\leq\; \int_0^{\max\{N(y_{2n},y_{2n+1},t),N(y_{2n+1},y_{2n+2},t)\}} \varphi(t)dt.$$

Similarly, we can obtain

$$\int_0^{M(y_{2n+2},y_{2n+3},kt)} \varphi(t)dt \;\geq\; \int_0^{\min\{M(y_{2n+1},y_{2n+2},t),M(y_{2n+2},y_{2n+3},t)\}} \varphi(t)dt,$$

$$\int_0^{N(y_{2n+2},y_{2n+3},kt)} \varphi(t)dt \;\leq\; \int_0^{\max\{N(y_{2n+1},y_{2n+2},t),N(y_{2n+2},y_{2n+3},t)\}} \varphi(t)dt.$$

In general,

$$\int_0^{M(y_{n+1},y_{n+2},kt)} \varphi(t)dt \;\geq\; \int_0^{\min\{M(y_n,y_{n+1},t),M(y_{n+1},y_{n+2},t)\}} \varphi(t)dt,$$

$$\int_0^{N(y_{n+1},y_{n+2},kt)} \varphi(t)dt \;\leq\; \int_0^{\max\{N(y_n,y_{n+1},t),N(y_{n+1},y_{n+2},t)\}} \varphi(t)dt.$$

and, for every positive integer p,

$$\int_0^{M(y_{n+1},y_{n+2},kt)} \varphi(t)dt \;\geq\; \int_0^{\min\{M(y_n,y_{n+1},t),M(y_{n+1},y_{n+2},t/k^p)\}} \varphi(t)dt,$$

$$\int_0^{N(y_{n+1},y_{n+2},kt)} \varphi(t)dt \;\leq\; \int_0^{\max\{N(y_n,y_{n+1},t),N(y_{n+1},y_{n+2},t/k^p)\}} \varphi(t)dt.$$

since $M(y_{n+1}, y_{n+2}, t/k^p) \to 1 \;\; as \;\; p \to \infty$, $N(y_{n+1}, y_{n+2}, t/k^p) \to 0 \;\; as \;\; p \to \infty$,

$$\int_0^{M(y_{n+1},y_{n+2},kt)} \varphi(t)dt \;\geq\; \int_0^{M(y_n,y_{n+1},t)} \varphi(t)dt.$$

$$\int_0^{N(y_{n+1},y_{n+2},kt)} \varphi(t)dt \;\; \leq \;\; \int_0^{N(y_n,y_{n+1},t)} \varphi(t)dt.$$

By Lemma 2.13, $\{y_n\}$ is Cauchy sequence in X. Since X is a complete, there is a point z in X such that $y_n \to z \in X$. Hence from (3.3), we have

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z,$$

$$y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z.$$

Since A and S are compatible and reciprocally continuous mappings, then $ASx_{2n} \to Az$ and $SAx_{2n} \to Sz$ as $n \to \infty$. The compatibility of the pair (A, S) yields

$$\lim_{n\to\infty} M(ASx_{2n}, SAx_{2n}, t) = 1$$

That is,

$$M(Az, Sz, t) = 1. \text{ Hence } Az = Sz.$$

The compatibility of the pair (A, S) yields

$$\lim_{n\to\infty} N(ASx_{2n}, SAx_{2n}, t) = 0$$

That is,

$$N(Az, Sz, t) = 0. \text{ Hence } \; Az = Sz.$$

Since $AX \subset TX$, there exist $w \in X$ such that $Az = Tw$. Using (ii), we get

$$\int_0^{M(Az,Bw,kt)} \varphi(t)dt \;\; \geq \;\; \int_0^{m(z,w,t)} \varphi(t)dt,$$

$$\int_0^{N(Az,Bw,kt)} \varphi(t)dt \;\; \leq \;\; \int_0^{n(z,w,t)} \varphi(t)dt,$$

Take $\alpha = 1$,

$$\begin{aligned} m(z,w,t) &= \min\{M(Tw,Bw,t), M(Sz,Az,t), M(Sz,Bw,t), \\ &\quad M(Tw,Az,t), M(Tw,Sz,t)\} \\ &= \min\{M(Az,Bw,t), 1, M(Az,Bw,t), 1, 1\} \\ &= \min\{M(Az,Bw,t), 1\}, \end{aligned}$$

$$\begin{aligned} n(z,w,t) &= \max\{N(Tw,Bw,t), N(Sz,Az,t), N(Sz,Bw,t), \\ &\quad N(Tw,Az,t), N(Tw,Sz,t)\} \\ &= \max\{N(Az,Bw,t), 1, N(Az,Bw,t), 1, 1\} \\ &= \max\{N(Az,Bw,t), 1\}. \end{aligned}$$

$$\int_0^{M(Az,Bw,kt)} \varphi(t)dt \geq \int_0^{M(Az,Bw,t)} \varphi(t)dt,$$

$$\int_0^{N(Az,Bw,kt)} \varphi(t)dt \leq \int_0^{N(Az,Bw,t)} \varphi(t)dt.$$

By using Lemma 2.14, we get $Az = Bw$. Thus,

$$Sz = Az = Bw = Tw.$$

Pointwise R-weakly commuting of A and S implies that there exists $R > 0$ such that

$$M(ASz, SAz, t) \geq M(Az, Sz, t/R) = 1,$$
$$N(ASz, SAz, t) \leq N(Az, Sz, t/R) = 0.$$

That is,

$$ASz = SAz \quad \text{and} \quad AAz = ASz = SAz = SSz.$$

Similarly, Pointwise R-weakly commuting of B and T implies that there exists $R > 0$ such that

$$M(BTw, TBw, t) \geq M(Bw, Tw, t/R) = 1,$$

$$N(BTw, TBw, t) \leq N(Bw, Tw, t/R) = 0.$$

That is,

$$BTw = TBw \quad \text{and} \quad BBw = BTw = TBw = TTw.$$

Using (ii), we get

$$\begin{aligned}
\int_0^{M(Az,AAz,kt)} \varphi(t)dt &= \int_0^{M(AAz,Bw,kt)} \varphi(t)dt, \\
&\geq \int_0^{m(Az,w,t)} \varphi(t)dt, \\
&= \int_0^{M(Az,AAz,t)} \varphi(t)dt,
\end{aligned}$$

$$\begin{aligned}
\int_0^{N(Az,AAz,kt)} \varphi(t)dt &= \int_0^{N(AAz,Bw,kt)} \varphi(t)dt, \\
&\leq \int_0^{n(Az,w,t)} \varphi(t)dt, \\
&= \int_0^{N(Az,AAz,t)} \varphi(t)dt.
\end{aligned}$$

By using Lemma 2.14, we get $Az = AAz$ and $Az = AAz = SAz$. Thus, Az is a common fixed point of A and S. Similarly, by using (ii), we get Bw (=Az) is a common fixed point of B and T. Uniqueness of the common fixed point follows easily and the proof is similar when B and T are assumed compatible and reciprocally continuous. □

Example 3.2. *Let $X = [2, 20]$ and $(X, M, N, *, \diamondsuit)$ be a intuitionistic fuzzy metric. Define mappings $A, B, S, T : X \to X$ by*

$$A(x) = \begin{cases} 2 & \text{if } x = 2, \\ 3 & \text{if } x > 2. \end{cases}$$

$$S(x) = \begin{cases} 2 & \text{if } x = 2, \\ 6 & \text{if } x > 2. \end{cases}$$

$$B(x) = \begin{cases} 2 & \text{if } x = 2 \text{ or } > 5, \\ 6 & \text{if } 2 < x \leq 5. \end{cases}$$

$$T(x) = \begin{cases} 2 & \text{if } x = 2, \\ 12 & \text{if } 2 < x \leq 5, \\ x - 3 & \text{if } x > 5. \end{cases}$$

Also, we Define,

$$M(Ax, By, t) = \frac{t}{(t + |x - y|)]}, N(Ax, By, t) = \frac{|x - y|}{(t + |x - y|)},$$

for all $x, y \in X$, $t > 0$. Then A, B, S and T satisfy all the conditions of the above Theorem with $k = (0, 1)$ and $\varphi(t) = 1$ and have a unique common fixed point $x = 2$. Here, A and S are reciprocally continuous compatible maps. But neither A nor S is continuous, even at the common fixed point $x = 2$. The mapping B and T are non-compatible but pointwise R-weakly commuting. B and T are pointwise R-weakly commuting since they commute at their coincidence points. To see that B and T are non-compatible, let us consider the sequence $\{x_n\}$ defined by

$$x_n = 5 + 1/n, n \geq 1. \text{ Then } Tx_n \to 2, Bx_n = 2, TBx_n = 2, BTx_n = 6.$$

Hence B and T are noncompatible.

Remark 3.3. *All the mappings involved in this example are discontinuous at the common fixed point.*

Remark 3.4. *Compatible maps are necessarily pointwise R weakly commuting since compatible maps commute at their coincidence points. However, as shown in the above example for the mappings B and T, pointwise R- weakly commuting maps need not be compatible.*

Remark 3.5. *In this remark we demonstrate that pointwise R-weak commutativity is a necessary condition for the existence of common fixed points of contractive mapping pairs. So, let us assume that the self mappings A and S of an intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ satisfy the contractive condition*

$$\int_0^{M(Ax,Ay,kt)} \varphi(t)dt \;>\; \int_0^{m(x,y,t)} \varphi(t)dt,$$

where

$$\begin{aligned} m(x,y,t) &= \min\{M(Sx,Sy,t), M(Ax,Sx,t), M(Ay,Sy,t), \\ &\quad M(Ax,Sy,t), M(Ay,Sx,t)\}, \end{aligned}$$

$$\int_0^{N(Ax,Ay,kt)} \varphi(t)dt \;<\; \int_0^{n(x,y,t)} \varphi(t)dt,$$

where

$$\begin{aligned} n(x,y,t) &= \max\{N(Sx,Sy,t), N(Ax,Sx,t), N(Ay,Sy,t), \\ &\quad N(Ax,Sy,t), N(Ay,Sx,t)\}. \end{aligned}$$

which is one of the general contractive definitions for a pair of mappings. If possible, suppose that A and S fail to be pointwise R-weakly commuting and yet have a common fixed point z. Then $z = Az = Sz$ and there exists x in X such that $Ax = Sx$ but $ASx \neq SAx$. clearly, $z \neq x$ since $ASz = SAz = z$. Moreover, $Ax \neq Az$. But then we have

$$\int_0^{M(Ax,Az,kt)} \varphi(t)dt \;>\; \int_0^{m(x,z,t)} \varphi(t)dt,$$

where

$$\begin{aligned} m(x,z,t) &= \min\{M(Sx,Sz,t), M(Ax,Sx,t), M(Ay,Sz,t), \\ &\quad M(Ax,Sz,t), M(Az,Sx,t)\} \\ &= M(Ax,Az,t) \end{aligned}$$

$$\int_0^{N(Ax,Ay,kt)} \varphi(t)dt \quad < \quad \int_0^{n(x,y,t)} \varphi(t)dt,$$

where

$$\begin{aligned} n(x,z,t) &= \max\{N(Sx,Sz,t), N(Ax,Sz,t), N(Az,Sz,t), \\ &\quad N(Ax,Sz,t), N(Ay,Sz,t)\} \\ &= N(Ax,Az,t) \end{aligned}$$

$$\int_0^{M(Ax,Az,kt)} \varphi(t)dt \quad > \quad \int_0^{M(Ax,Az,t)} \varphi(t)dt,$$

$$\int_0^{N(Ax,Az,kt)} \varphi(t)dt \quad < \quad \int_0^{N(Ax,Az,t)} \varphi(t)dt,$$

a contradiction. Hence the assertion.

References

[1] Alaca.C, Turkoglu.D, Yildiz.C, Fixed points in intuitionistic fuzzy metric spaces, *Chaos, Solitons and Fractals*, (2006), 29, 1073-1078.

[2] A.Aliouche, 'A Common fixed point Theorem for symmetric spaces satisfying a contractive condition of condition of integral type', *Journal of Mathematics Analysis and Application*, 322 (2006), 796-802.

[3] Atanassov.k, Intuitionistic fuzzy sets, *Fuzzy sets and system*, 20 (1986), 87-96.

[4] Atanassov.k, New operations defined over the intuitionistic fuzzy sets, *Fuzzy sets and system*, 61 (1994),137-142.

[5] Bijendra Singh and M.S. Chauhan, 'Common fixed points of compatible maps in fuzzy metric spaces', *Fuzzy sets and system*, 115 (2000), 471-475.

[6] A.Branciari, 'A fixed point theorem for mappings satisfying a general contractive condition of integral type', *International Journal of Mathematics and Mathematical Science*, 29 (2002), 531-536.

[7] Coker.D, An introduction to intuitionistic fuzzy topological spaces, *Fuzzy sets and system*, 88 (1997), 81-89.

[8] A.George and Veeramani, 'On some results in fuzzy metric spaces', *Fuzzy sets and systems*, 64 (1994),64, 395-399.

[9] Gregori.V, Sapena.A, 'On Fixed point theorem in fuzzy metric spaces', *Fuzzy sets and systems*, 125 (2002), 245-252.

[10] Hadzic.O, Fixed Point Theory in probabilistic meric spaces, Navi Sad: Serbian Academy of Science and arts,1995.

[11] G.Jungck, 'Compatible Mappings and common fixed points', *International Journal of Mathematics and Mathematical Science.*, 9 (1986), 771-79.

[12] O.Kramosil and J.Michalek, Fuzzy metric and Statiscal metric spaces, Kybernetica, 11 (1975), 326-334.

[13] O.Kaleva and S.Seikkala, On Fuzzy metric spaces, *Fuzzy sets and systems*, 12 (1984), 215-229.

[14] R.P.Pant, 'Common Fixed Points of Noncommuting Mappings', *Journal of Mathematical Analysis and Applications*, 188 (1994), 436-440.

[15] R.P.Pant, 'Common Fixed Points of Four Mappings', *Bulletin of Calcutta Mathematical Society*, 90 (1998), 281-286.

[16] Park JH, Intuitionistic fuzzy metric spaces, *Chaos, Solitons and Fractals*, 22 (2004), 1039-1046.

[17] Pathak H.K, Cho Y.J and Kang S.M., 'Remarks on R-weakly commuting mappings and common fixed point theorems', *Bullentin of Korean Mathematical Society*, 34 (2), 1997, 247-257.

[18] B.Schweizer and A.Sklar, Statistical spaces, Pacific Journal of Mathematics, 10(1960), 313-334.

[19] Sedhi.S, Shobe.N, A.Aliouche, 'Common fixed point theorems in intuitionistic fuzzy metric spaces through conditions of integral type' , *Applied Mathematics and Information sciences*, 2(1) (2008),61-82.

[20] Sushil Sharma, Servet Kutukcu and R.S Rathore, 'Common fixed point for Multivalued mappings in intuitionistic fuzzy metric space', *Communication of Korean Mathematical Society*, 22 (3),(2007), 391-399.

[21] L.A.Zadeh, Fuzzy Sets, Information Control, 89 (1965), 338-353.

Received: August, 2008