Continuous Multi-Utility for
Extremely Continuous Preorders

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Abstract

We present some sufficient conditions for the existence of a continuous multi-utility representation of a preorder $\succeq$ on a topological space $(X, \tau)$ (i.e., for the existence of a family $\mathcal{F}$ of continuous increasing real-valued functions on the preordered topological space $(X, \tau, \succeq)$ such that, for every pair of elements $(x, y) \in X \times X$, $x \succeq y \iff f(x) \leq f(y) \forall f \in \mathcal{F}$). Such conditions are based on the concept of an extremely continuous preorder on a topological space.

Mathematics Subject Classification: 06A06, 54F05

Keywords: continuous multi-utility representation, continuous utility function, extremely continuous preorder

1 Introduction

The (continuous) representation of a not necessarily total (complete) preorder $\succeq$ on a topological space $(X, \tau)$ has received a considerable attention in the literature since the seminal paper by Aumann [1], who pointed out the fact that there may exist two alternatives $x, y \in X$ which are “incomparable” (i.e., neither $x \succeq y$ nor $y \succeq x$) and the individual is not a
*priori* forced to declare that they are “indifferent” (in the sense that $x \preceq y$ and $y \preceq x$).

Peleg [11] was the first who presented sufficient conditions for the existence of a continuous utility function for a *preorder* (or for a *partial order*) on a topological space.

We recall that if we denote by $(X, \preceq)$ a preordered set, then a function $u : (X, \preceq) \rightarrow (\mathbb{R}, \leq)$ is referred to as a *utility function à la Richter-Peleg* (see Peleg [11] and Richter [12]) if the following two conditions are verified for all $(x, y) \in X \times X$,

\begin{align*}
(i) \quad (x \preceq y) &\Rightarrow u(x) \leq u(y), \\
(ii) \quad (x \prec y) &\Rightarrow u(x) < u(y),
\end{align*}

where the symbol $\prec$ stands for the *strict part* of $\preceq$ (namely, for all $x, y \in X$, $x \prec y$ if and only if $x \preceq y$ and not($y \preceq x$)). A utility function à la Richter-Peleg is more frequently called an *order-preserving function* or simply a *utility function*.

If $X$ is endowed with a topology $\tau$ and we denote by $\tau_{nat}$ the *natural topology* on $\mathbb{R}$, then one may be interested in ensuring the existence of a continuous utility function à la Richter-Peleg $u : (X, \tau, \preceq) \rightarrow (\mathbb{R}, \tau_{nat}, \leq)$. However, the existence of such a utility function $u$ for a non total preorder $\preceq$ does not allow to recover the preorder $\preceq$ (i.e., to characterize it).

For this reason, other kinds of representations of non total preorders have been recently introduced. From Evren and Ok [7], a preorder $\preceq$ on a topological space $(X, \tau)$ is said to have a *continuous multi-utility representation* if there exists a family $\mathcal{F}$ of continuous increasing (isotone) real-valued functions on the preordered topological space $(X, \tau, \preceq)$ such that, for every pair of elements $(x, y) \in X \times X$,

$$x \preceq y \iff f(x) \leq f(y) \quad \forall f \in \mathcal{F}.$$  

In decision theory multi-utility representations have been mainly studied in connection with possible generalizations of expected utility representations of incomplete preferences (see e.g. Evren [6]). A (continuous) multi-utility representation is important since it characterizes numerically a not necessarily total preorder. It is worthwhile noticing that Ok [10] furnished another important contribution in this direction by providing conditions under which there exists a (continuous) utility function $u : X \rightarrow \mathbb{R}^n$ representing a preorder $\preceq$ on $X$, in the sense that, for all $x, y \in X$,

$$x \preceq y \text{ if and only if } u(x) \leq_{prod} u(y),$$
where $\leq_{\text{prod}}$ is the usual \textit{product preorder} on $\mathbb{R}^n$.

Evren and Ok [7] provided some conditions for the existence of an upper semicontinuous or continuous multi-utility representation for a preorder on a topological space. The authors proved that there exists an upper semicontinuous multi-utility representation for every upper semicontinuous preorder $\preceq$ on a topological space $(X, \tau)$, where a preorder $\preceq$ is said to be “upper semicontinuous” if $U_{\preceq}(x) = \{z \in X : x \preceq z\}$ is a closed subset of $X$ for every $x \in X$. On the other hand, Evren and Ok showed that the problem of finding a continuous multi-utility representation for a preorder on a topological space has not a simple solution in general.

In this paper we present sufficient conditions for the existence of a continuous multi-utility representation by using the concept of an extremely continuous preorder on a topological space. From Mashburn [9], a preorder $\preceq$ on a topological space $(X, \tau)$ is said to be extremely continuous if every decreasing (increasing) subset of $X$ that has a maximal (minimal) element is closed. We use some of the interesting results presented by Mashburn [9] and some classical results concerning the existence of extensions by means of linear (pre)orders (see Dushnik and Miller [5] and Szpilrajn [14]) in order to guarantee the existence of a continuous multi-utility representation for a preorder on a topological space. To this aim, the Souslin hypothesis is particularly helpful.

2 Notation and preliminaries

A topology $\tau$ on a set $X$ is said to have the \textit{continuous representability property} (see e.g. Campión et al. [4]) if every continuous total preorder $\preceq$ on $(X, \tau)$ has a continuous utility representation (i.e., there exists a continuous real-valued function $f$ on $X$ such that, for all $x, y \in X$, $x \preceq y$ if and only if $f(x) \leq f(y)$). We recall that such a concept was first introduced by Herden [8], who referred to a topology with the indicated property as a \textit{useful topology}.

A topology $\tau$ on a set $X$ is said to have the \textit{semicontinuous representability property} if every upper semicontinuous total preorder $\preceq$ on $(X, \tau)$ has an upper semicontinuous utility representation (i.e., there exists an upper semicontinuous real-valued function $f$ on $X$ such that, for all $x, y \in X$, $x \preceq y$ if and only if $f(x) \leq f(y)$). A topology of this kind is referred to as a \textit{completely useful topology} by Bosi and Herden [3].

Further, recall that a preorder $\preceq$ on a topological space $(X, \tau)$ is said to be extremely continuous if every decreasing (increasing) subset of $X$ that has a maximal (minimal) element is closed. It is clear that a total preorder $\preceq$ on a topological space $(X, \tau)$ is extremely continuous if and only if it is continuous (in the sense that the sets $L_{\preceq}(x) = \{z \in X : z \preceq x\}$ and $U_{\preceq}(x) = \{z \in X : x \preceq z\}$ are closed subsets of $X$ for every $x \in X$).

On the other hand, a preorder $\preceq$ on a topological space $(X, \tau)$ is said to
be upper (lower) semicontinuous if \( U_{\leq}(x) = \{ z \in X : x \leq z \} \) \( (L_{\geq}(x) = \{ z \in X : z \geq x \}) \) is a closed subset of \( X \) for every \( x \in X \).

A preorder \( \preceq \) is said to be an extension of a preorder \( \preceq' \) on a set \( X \) if \( \prec \subseteq \prec' \) and \( \prec \subseteq \prec' \). It is well known that the classical Szpilrajn theorem guarantees that every (pre)order may be extended to a total (pre)order.

From Evren and Ok [7], a preorder (i.e., a reflexive and transitive binary relation) \( \preceq \) on a topological space \( (X, \tau) \) is said to have a continuous multi-utility representation if there exists a family \( F \) of continuous increasing (isotone) real-valued functions on the preordered topological space \( (X, \tau, \preceq) \) such that, for every pair of elements \( (x, y) \in X \times X \),

\[
x \preceq y \iff f(x) \leq f(y) \quad \forall f \in F.
\]

(1)

3 Sufficient conditions for continuous multi-utility

In order to prove the following propositions, we shall use a classical and well known theorem by Dushnik and Miller (see Dushnik and Miller [5, Theorem 2.32]) and the results due to Mashburn [9] concerning both the existence of continuous linear extensions of a preorder and the notion of a pliable space (see in particular Mashburn [9, Lemma 8, Corollary 13]).

**Proposition 3.1** Let \( \preceq \) be an extremely continuous preorder on a topological space \( (X, \tau) \). If \( \tau \) has the continuous representability property, then \( \preceq \) has a continuous multi-utility representation.

**Proof.** Since \( \preceq \) is an extremely continuous preorder on \( (X, \tau) \), we have that every extension of \( \preceq \) is continuous (see Mashburn [9, Lemma 8])). Obviously, every linear extension \( \preceq' \) of \( \preceq \) is continuous and therefore it is representable by a continuous utility function due to the fact that the topology \( \tau \) on \( X \) has the continuous representability property. Hence, from Dushnik and Miller [5, Theorem 2.32], there exists a collection \( K \) of linear preorders such that, for every pair \( (x, y) \in X \times X \),

\[
x \preceq y \iff x \preceq y \quad \forall \preceq' \in K \iff f(x) \leq f(y) \quad \forall f \in F,
\]

where \( F \) is the collection of all the continuous utility functions for the linear preorders in \( K \). \( \square \)

**Remark 3.2** As an immediate consequence of Proposition 3.1, we have that an extremely continuous preorder \( \preceq \) on a topological space \( (X, \tau) \) has a continuous multi-utility representation provided that \( (X, \tau) \) is either second
countable (see the continuous utility representation theorem of Debreu) or connected and separable (see to the continuous utility representation theorem of Eilenberg-Debreu) or else, for example, locally connected and separable (see Campion et al. [4, Corollary 3.4]).

From Bosi and Herden [3], a topology \( \tau \) on a set \( X \) is said to be \textit{countably isolated} if every family \( O \) of open subsets of \( X \) which is linearly ordered by set inclusion only contains countably many isolated sets, where a set \( O \in O \) is said to be \textit{isolated} if \( \bigcup_{O \supset O' \supset O'' \supset \cdots} O'' \not\subseteq O \subseteq \cdots \not\subseteq O' \subseteq O \). The following corollary is an immediate consequence of Proposition 3.1 on one hand, and on the other hand of Proposition 4.4 and Theorem 4.8 in Bosi and Herden [3].

We recall that the Souslin hypothesis states that that every chain \((Z, \leq)\) that satisfies \textit{ccc} and only has countably many jumps can be order-embedded into the real line. A chain (i.e., a linearly ordered set) \((Z, \leq)\) satisfies the \textit{countable chain condition (ccc)} if every family of pairwise disjoint open intervals of \((Z, \leq)\) is countable.

**Corollary 3.3** Under the Souslin hypothesis, if \( \preceq \) is an extremely continuous preorder on a topological space \((X, \tau)\) and \( \tau \) is countably isolated, then \( \preceq \) has a continuous multi-utility representation.

**Proof.** We have just to observe that \( \tau \) has the semicontinuous representability property since it is countably isolated and the Souslin hypothesis holds (see Theorem 4.8 in Bosi and Herden [3]). Then we have that \( \tau \) has the continuous representability property by Proposition 4.4 in Bosi and Herden [3] and therefore we are now ready to apply Proposition 3.1. \( \square \)

Before presenting the statement of the following proposition, let us recall that from Mashburn [9] a collection \( A \) of subsets of a preordered set \((X, \preceq)\) is said to be a \textit{collection of nonoverlapping subsets} of \( X \) if the following conditions are satified:

1. elements of \( A \) are pairwise disjoint and every element of \( A \) has at least two elements;
2. the transitive closure of the relation \( \{(A, B) \in A^2 : A \neq B \text{ and } \exists a \in A \exists b \in B (a \prec b)\} \) is an order;
3. for every \( A, B \in A \), if \( a \in A \), \( b \in B \) and \( a \neq b \) then \( a \not\sim b \).

We further recall that a preordered topological space \((X, \tau, \preceq)\) is said to be \textit{pliable} if there exists a continuous utility function \( f \) à la Richter-Peleg for every extension \( \succeq \) of \( \preceq \) (see the introduction).
Proposition 3.4 Under the Souslin hypothesis, if $\preceq$ is an extremely continuous preorder on a topological space $(X, \tau)$ and every collection $\mathcal{A}$ of nonoverlapping subsets of $X$ is at most countable, then $\preceq$ has a continuous multi-utility representation.

Proof. Since the Souslin hypothesis holds, $\preceq$ is an extremely continuous preorder on $(X, \tau)$ and every collection $\mathcal{A}$ of nonoverlapping subsets of $X$ is at most countable, we have that from Mashburn [9, Corollary 13] the preordered topological space $(X, \tau, \preceq)$ is pliable and therefore every (linear) extension $\lessgtr$ of $\preceq$ is representable by a continuous utility function. Therefore, the thesis follows from the proof of Proposition 3.1 since the famous Szpilrajn theorem allows us to only consider the linear extensions of the preorder $\preceq$.

References


Received: September, 2008