On Total Dominating Sets in Graphs

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Abstract

A set $S$ of vertices in a graph $G(V,E)$ is called a dominating set if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. A set $S$ of vertices in a graph $G(V,E)$ is called a total dominating set if every vertex $v \in V$ is adjacent to an element of $S$. The domination number of a graph $G$ denoted by $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. Respectively the total domination number of a graph $G$ denoted by $\gamma_t(G)$ is the minimum cardinality of a total dominating set in $G$. An upper bound for $\gamma_t(G)$ which has been achieved by Cockayne and et al. in [1] is: for any graph $G$ with no isolated vertex and maximum degree $\Delta(G)$ and $n$ vertices, $\gamma_t(G) \leq n - \Delta(G) + 1$.

Here we characterize bipartite graphs and trees which achieve this upper bound. Further we present some another upper and lower bounds for $\gamma_t(G)$. Also, for circular complete graphs, we determine the value of $\gamma_t(G)$.

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1 Introduction

Let $G(V,E)$ be a graph. For any vertex $x \in V$, we define the neighborhood of $x$, denoted by $N(x)$, as the set of all vertices adjacent to $x$. The closed neighborhood of $x$, denoted by $N[x]$, is the set $N(x) \cup \{x\}$. For a set of vertices $S$, we define $N(S)$ as the union of $N(x)$ for all $x \in S$, and $N[S] = N(S) \cup S$. The degree of a vertex is the size of its neighborhoods. The maximum degree of a graph $G$ is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. Here $n$ will denote the number of vertices of a graph $G$. A set $S$ of vertices in a graph $G(V,E)$ is called a dominating set if every vertex $v \in V$ is either
an element of $S$ or is adjacent to an element of $S$. A set $S$ of vertices in a
graph $G(V, E)$ is called a total dominating set if every vertex $v \in V$ is adjacent
to an element of $S$. The domination number of a graph $G$ denoted by $\gamma(G)$
is the minimum cardinality of a dominating set in $G$. Respectively the total
domination number of a graph $G$ denoted by $\gamma_t(G)$ is the minimum cardinality
of a total dominating set in $G$. Clearly $\gamma(G) \leq \gamma_t(G)$, also it has been proved
that $\gamma_t(G) \leq 2\gamma(G)$.

An upper bound for $\gamma_t(G)$ has been achieved by Cockayne and et al. in [1] in
the following theorems:

**THEOREM A** If a graph $G$ has no isolated vertices, then $\gamma_t(G) \leq n - 
\Delta(G) + 1$.

**THEOREM B** If $G$ is a connected graph and $\Delta(G) < n - 1$, then $\gamma_t(G) \leq
n - \Delta(G)$

As a result of the above theorems, if $G$ is a graph with $\gamma_t(G) = n - \Delta(G) + 1,$
then $\Delta(G) \geq n - 1$. Hence, if $G$ is a $k$-regular graph and $\gamma_t(G) = n - k + 1,$
then $G$ is $K_n$. As a result of the above theorems, if $G$ is a graph with $\gamma_t(G) =
n - \Delta(G) + 1,$ then $\Delta(G) \geq n - 1$. Hence, if $G$ is a $k$-regular graph and
$\gamma_t(G) = n - k + 1,$ then $G$ is $K_n$. Total domination and upper bounds on the
total domination number in graphs were intensively investigated, see e. g. ([3], [4]).

Here we characterize bipartite graphs and trees which achieve the upper bound
in Theorem A. Further we present some another upper and lower bounds for
$\gamma_t(G)$. Also, for circular complete graphs, we determine the value of $\gamma_t(G)$.

It is easy to prove that for $n \geq 3$, $\gamma_t(C_n) = \gamma_t(P_n) = \frac{n}{2}$ if $n \equiv 0 \pmod{4}$
and $\gamma_t(C_n) = \gamma_t(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ otherwise.

for the definitions and notations not defined here we refer the reader to
texts, such as [2].

### 2 Other bounds for $\gamma_t(G)$

In this section we introduce some other upper bounds for $\gamma_t(G)$.

**Theorem 2.1** Let $G$ be a connected graph, then $\gamma_t(G) \geq \left\lceil \frac{n}{\Delta(G)} \right\rceil$.

**Proof:** Let $S \subseteq V(G)$ be a total dominating set in $G$. Every vertex in $S$
dominates at most $\Delta(G) - 1$ vertices of $V(G) - S$ and dominate at least one
of the vertices in $S$. Hence, $|S|((\Delta(G) - 1) + |S|) \geq n$. Since, $S$ is an arbitrary
total dominating set, then $\gamma_t(G) \geq \left\lceil \frac{n}{\Delta(G)} \right\rceil$.

If $G = K_n$, $G = C_{4n}$, or $G = P_{4n}$ then $\gamma_t(G) = \left\lceil \frac{n}{\Delta(G)} \right\rceil$, so the above bound is sharp.
Theorem 2.2 Let $G$ be a graph with $\text{diam}(G) = 2$ then, $\gamma_t(G) \leq \delta(G) + 1$.

**Proof:** Let $x \in V(G)$ and $\deg(x) = \delta(G)$. Since, $\text{diam}(G) = 2$, then $N(x)$ is a dominating set for $G$. Now $S = N(x) \cup \{x\}$ is a total dominating set for $G$ and $|S| = \delta(G) + 1$. Hence, $\gamma_t(G) \leq \delta(G) + 1$. \hfill $\blacksquare$

As we know, $\gamma_t(C_5) = 3$ and also $\delta(C_5) = 2$, $\text{diam}(C_5) = 2$ then $\gamma_t(C_5) = \delta(C_5) + 1$. Hence, the above bound is sharp.

**Theorem 2.3** If $G$ is a connected graph with the girth of length $g(G) \geq 5$ and $\delta(G) \geq 2$, then $\gamma_t(G) \leq n - \lceil \frac{g(G)}{2} \rceil + 1$.

**Proof:** Let $G$ be a connected graph with $g(G) \geq 5$ and let $C$ be a cycle of length $g(G)$. Remove $C$ from $G$ to form a graph $G'$. Suppose an arbitrary vertex $v \in V(G')$, since $\delta(G) \geq 2$, then $v$ has at least two neighbors say $x$ and $y$. Let $x, y \in C$. If $d(x, y) \geq 3$, then replacing the path from $x$ to $y$ on $C$ with the path $x, v, y$ reduces the girth of $G$, a contradiction. If $d(x, y) \leq 2$, then $x, y, v$ are on either $C_3$ or $C_4$ in $G$, contradicting the hypothesis that $g(G) \geq 5$. Hence, no vertex in $G'$ has two or more neighbors on $C$. Since $\delta(G) \geq 2$, the graph $G'$ has minimum degree at least $\delta(G) - 1 \geq 1$. Then $G'$ has no isolated vertex. Now let $S'$ be a $\gamma_t$-set for $C$. Then $S = S' \cup V(G')$ is a total dominating set for $G$. Hence, $\gamma_t(G) \leq n - \lceil \frac{g(G)}{2} \rceil + 1$ (note that $\gamma_t(C) \leq \lceil \frac{g(G)}{2} \rceil + 1$). \hfill $\blacksquare$

## 3 Bipartite graphs with $\gamma_t(G) = n - \Delta(G) + 1$

In this section we characterize the bipartite graphs achieving the upper bound in the theorem A.

**Theorem 3.4** Let $G$ be a bipartite graph with no isolated vertices. Then $\gamma_t(G) = n - \Delta(G) + 1$ if and only if $G$ is a graph in form of $K_{1,t} \cup rK_2$ for $r \geq 0$.

**Proof:** If $G$ is $K_{1,t} \cup rK_2$ ($r \geq 0$), clearly $\gamma_t(G) = n - \Delta(G) + 1$. Now let $G$ be a bipartite graph with partitions $A \cup B$ and $x \in A$ where $\deg(x) = \Delta(G) = t$. We continue our proof in four stages:

**Stage 1:** We claim that for every vertex $y \in A - \{x\}$, $N(y) - N(x) \neq \emptyset$. If it is not true, there exists a vertex in $A - \{x\}$, say $y$, such that $N(y) \subseteq N(x)$. So let $u \in N(y)$, the set $S = V - (N(x) \cup \{y\}) \cup \{u\}$ is a total dominating set and $|S| = n - \Delta(G)$, a contradiction. So we have $n \geq 2|A| + \Delta(G) - 1$.

**Stage 2:** For every vertex $y \in A$, let $u_y \in N(y)$. Clearly the set $S = A \cup \bigcup_{y \in A} \{u_y\}$ is a total dominating set for $G$ and $|S| \leq 2|A|$, so $\gamma_t(G) \leq 2|A|$. Now let $y \in A - \{x\}$ such that $|N(y) - N(x)| \geq 2$. Hence, we have:
\[ n \geq 2|A| + \Delta(G) \]
\[ \Rightarrow \gamma_t(G) + \Delta(G) - 1 \geq 2|A| + \Delta(G) \]
\[ \Rightarrow \gamma_t(G) \geq 2|A| + 1, \]

a contradiction. Hence, for every vertex \( y \in A - \{x\}, |N(y) - N(x)| = 1. \)

**Stage 3:** Let \( y \in A - \{x\} \) and \( N(y) \cap N(x) \neq \emptyset \). Let \( u \in N(y) \cap N(x) \). Now, \( S = (V - N(x) \cup \{y\}) \cup \{u\} \) is a total dominating set and \( |S| = n - \Delta(G) \).

So, \( \gamma_t(G) \leq n - \Delta(G), \) a contradiction.

**Stage 4:** Let \( y, z \in A - \{x\} \) and \( N(y) \cap N(z) \neq \emptyset \). Now \( S = (V - (\{z\} \cup N(x))) \cup \{u\} \), where \( u \in N(x) \), is a total dominating set and \( |S| = n - \Delta(G) \).

So, \( \gamma_t(G) \leq n - \Delta(G) \), a contradiction. Hence, \( G \) is a graph in form of \( K_{1,t} \cup rK_2. \)

**Corollary 3.1** Let \( T \) is a Tree. Then \( \gamma_t(T) = n - \Delta(T) + 1 \) if and only if \( T \) is a star.

## 4 Total domination numbers of circular complete graphs

If \( n \) and \( d \) are positive integers with \( n \geq 2d \), then circular complete graph \( K_{n,d} \) is the graph with vertex set \( \{v_0, v_1, \ldots, v_{n-1}\} \) in which \( v_i \) is adjacent to \( v_j \) if and only if \( d \leq |i - j| \leq n - d \). In this section we determine the total domination of circular complete graphs. It is easy to see that \( K_{n,1} \) is the complete graph \( K_n \) and \( K_{n,2} \) is a circle on \( n \) vertices, therefore we assume that \( d \geq 3. \)

**Theorem 4.5** For \( n \geq 4d - 2 \) and \( d \geq 3 \), \( \gamma_t(K_{n,d}) = 2. \)

**Proof:** Clearly, \( \gamma_t(K_{n,d}) \geq 2. \) Let \( S = \{v_0, v_{2d-1}\} \). We will show that \( S \) is a total dominating set for \( K_{n,d} \). Since \( n \geq 4d - 2 \) and \( 2d - 1 \leq 2d \), then \( 2d - 1 \leq n - d \). Also \( 2d - 1 \geq d \) since \( d \geq 3 \). Thus \( d \leq 2d - 1 \leq n - d \) and \( v_0v_{2d-1} \in E(K_{n,d}) \). By definition of \( K_{n,d} \), \( v_0 \) is adjacent to each of the vertices \( v_d, v_{d+1}, \ldots, v_{n-d}. \)

Now for each \( 1 \leq i \leq d - 1 \) we have
\[ n - d + i - (2d - 1) = n - 3d + i + 1 \geq 4d - 2 - 3d + i + 1 \geq d \]

and
\[ n - d + i - (2d - 1) = n - 3d + i + 1 \leq n - 3d + d = n - 2d < n - d. \]
Thus \( v_{2d-1} \) is adjacent to each of the vertices \( v_{n-d+1}, \ldots, v_{n-1} \). On the other hand, for each \( 1 \leq i \leq d-1 \) we have

\[
2d - 1 - i \leq 2d - 2 \leq 3d - 2 \leq n - d
\]

and

\[
2d - 1 - i \geq 2d - 1 - d + 1 = d.
\]

Hence \( v_{2d-1} \) is adjacent to each of the vertices \( v_0, v_1, \ldots, v_{d-1} \) and so \( S \) is a total dominating set for \( K_{n,d} \) and \( \gamma_t(K_{n,d}) = 2 \).

**Theorem 4.6** For \( 3d \leq n \leq 4d - 3 \) and \( d \geq 3 \), \( \gamma_t(K_{n,d}) = 3 \).

**Proof:** Let \( S = \{v_0, v_d, v_{2d-1}\} \). We prove that \( S \) is a \( \gamma_t(K_{n,d}) \)-set. Since \( d \leq 2d - 2 \leq n - d \), \( G[S] \) contains no isolated vertices. Clearly \( v_0 \) and \( v_d \) are adjacent to each of the vertices \( v_d, v_{d+1}, \ldots, v_{n-d} \) and \( v_{2d}, v_{2d+1}, \ldots, v_{n-d} \) respectively. For \( 1 \leq i \leq d-1 \) we have

\[
2d - 1 - i \leq 2d - 1 - d + 1 = d
\]

and

\[
2d - 1 - i \leq 2d - 2 \leq 2d \leq n - d
\]

Thus \( v_{2d-1} \) is adjacent to each of the vertices \( v_1, v_2, \ldots, v_{d-1} \). Hence \( S \) is a total dominating set for \( K_{n,d} \) and so \( \gamma_t(K_{n,d}) \leq 3 \). Now we prove that there is no total dominating set for \( K_{n,d} \) of size 2. Let \( S' = \{u, v\} \) be a \( \gamma_t(K_{n,d}) \)-set. Without loss of generality, let \( u = v_0 \) and \( v = v_j \). Clearly \( d \leq j \leq n - d \). Since \( v_0 v_{n-d+1} \notin E(K_{n,d}) \), \( d \leq n - d + 1 - j \leq n - d \) and so \( 1 \leq j \leq d + 1 \). Thus \( j = d \) or \( j = d + 1 \). In both cases, \( S' \) is not a total dominating set since \( v_2, v_3, \ldots, v_{d-1} \) are not dominated by \( S' \) a contradiction. This completes the proof.

**References**


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