Helicoidal Surfaces in the Three-Dimensional Lorentz-Minkowski Space Satisfying $\Delta r_i = \lambda_i r_i$

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Abstract

In this paper, we classify helicoidal surfaces in the 3-dimensional Lorentz-Minkowski space under the condition $\Delta r_i = \lambda_i r_i$ where $\Delta$ is the Laplace operator with respect to the first fundamental form and $\lambda$ is a real number. We also give explicit forms of these surfaces.

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1. Introduction

Let $r : M^2 \to R^3_l$ be an isometric immersion of a helicoidal surface in the 3-dimensional Lorentz-Minkowski space equipped with the induced metric. By saying Lorentz-Minkowski space $R^3_l$, we mean the space $R^3$ with the standard metric given by

$$g = ds^2 = -dx^2 + dy^2 + dz^2$$

where $(x, y, z)$ is a rectangular coordinate system of $R^3_l$. Let $\Delta$ be the Laplace operator associated with the induced metric. Then a well known result due to Takahashi [6] states that minimal surfaces and spheres are the only surfaces in $R^3$ satisfying the condition

$$\Delta r = \lambda r , \; \lambda \in R$$

On the other hand Garay [3] determined the complete surfaces of revolu-
tion in $R^3$, whose component functions are eigenfunctions of their Laplace operator, i.e.

$$\Delta r^i = \lambda^i r^i, \quad \lambda^i \in R$$

Later the same author in [4] studied the hypersurfaces in $R^{n+1}$ for which

$$\Delta r = A r, \quad A \in R^{n+1} \times n+1$$

Recently in [1] Bekkar and Zoubir classified the surfaces of revolution with no zero Gaussian curvature $K_G$ in the 3-dimensional Lorentz-Minkowski space under the condition

$$\Delta x^i = \lambda^i x^i, \quad \lambda^i \in R$$

In [5] Kaimakamis and Papantoniou studied surfaces of revolution in the 3-dimensional Lorentz-Minkowski space satisfying the condition

$$\Delta^H r = A r,$$

where $\Delta^H$ is the Laplace operator with respect to the second fundamental form and $A$ is a real $3 \times 3$ matrix.

An interesting geometric question is raised, the classification of all helicoidal surfaces in 3-dimensional Lorentz-Minkowski space $R^3_l$, satisfying the condition

$$\Delta r_i = \lambda_i r_i \quad (1.1)$$

2. Preliminaries

Let $\gamma : I = (a, b) \subset R \rightarrow \Pi$ be a curve in a plane $\Pi$ of $R^3_l$ and let $\varepsilon$ be a straight line of $\Pi$ which does not intersect the curve $\gamma$. A helicoidal surface $M^2$ in $R^3_l$ is defined to be a non-degenerate surface which is generated by the rigid motions $g_t : R^3_l \rightarrow R^3_l, \ t \in R$ around the axis $\varepsilon$. In other words, a helicoidal surface $M^2$ with axis $\varepsilon$ in $R^3_l$ is invariant under the one parameter subgroup of rigid motions in $R^3_l$.

If the axis $\varepsilon$ is spacelike (resp. timelike), then there is a Lorentz transformation by which the axis $\varepsilon$ is transformed to the $z$–axis (resp. $x$–axis). Hence, without loss of generality, we may assume that the axis of revolution is the $z$–axis (resp. $x$–axis). If the axis is lightlike then we may suppose
that this is the line spanned by the vector \((1, 1, 0)\). Therefore we distinguish
the following three special cases.

**i-** Suppose that the axis of the revolution is the \(z\)-axis (spacelike) and
the curve \(\gamma\) is lying either in the \(y\) \(z\)-plane or in the \(x\) \(z\)-plane. Then a
parametrization of \(\gamma\), with respect to its arclength, is
\[ \gamma(u) = (0, f(u), g(u)) \]
and \(\gamma(u) = (f(u), 0, g(u))\) where \(f = f(u)\) is a positive function of classe
\(C^1\) and \(g = g(u)\) is a function of classe \(C^2\) on \(I = (a, b)\).

Hence, the helicoidal surface \(M^2\) given by [2] is defined by
\[
\begin{align*}
  r(u, v) &= (f(u) \sinh v, f(u) \cosh v, g(u) + cv) \\
  f(u) &> 0, \ c \in R^+.
\end{align*}
\] (1.2a)
or
\[
\begin{align*}
  r(u, v) &= (f(u) \cosh v, f(u) \sinh v, g(u) + cv) \\
  f(u) &> 0, \ c \in R^+.
\end{align*}
\] (1.2b)

We say that these helicoidal surfaces are of type I (Eq. (1.2a)) or II (Eq. (1.2b))
respectively.

**ii-** Suppose that the axis of the revolution is the \(x\)-axis (timelike) and
the curve \(\gamma\) is lying in the \(x\) \(y\)-plane. So, the curve \(\gamma\) is parametrized
by \(\gamma(u) = (g(u), f(u), 0)\). Hence, the helicoidal surface \(M^2\) is given by
[2]
\[
\begin{align*}
  r(u, v) &= (g(u) + cv, f(u) \cos v, f(u) \sin v) \\
  0 \leq v \leq 2\pi, \\
  f(u) &> 0, \ c \in R^+.
\end{align*}
\] (1.3)

This is called a helicoidal surface of type III.

**iii-** Suppose that the axis of revolution is a lightlike line, or equivalently
the line of the plane \(x\) \(y\) spanned by the vector \((1, 1, 0)\) and the curve \(\gamma\) lies in
the \(x\) \(y\)-plane. Then its parametrization is given by \(\gamma(u) = (f(u), g(u), 0), \ u \in I\)
where \(f\) and \(g\) are functions on \(I\), such that \(f(u) \neq g(u), \forall u \in I\).

Therefore the helicoidal surface \(M^2\) may be parametrized as
\[
\begin{align*}
  r(u, v) &= \left\{ \left(1 + \frac{v^2}{2}\right) f(u) - \frac{v^2}{2} g(u) + cv, \ \frac{v^2}{2} f(u) \\
  &\quad + \left(1 - \frac{v^2}{2}\right) g(u) + cv, \ f(u) - g(u) \right\}.
\end{align*}
\] (1.4)

This surface is called a helicoidal surface of type IV.

It should be noted that when \(c = 0\), the helicoidal surfaces in \(R^3\) are just
the surfaces of revolution. (For more details see [5]).

We say that a helicoidal surface \(M^2\) in \(R^3\) is of type I \(^+\) or \(I^−\) (resp.
\(II^+\) or \(II^−\), \(III^+\) or \(III^−\), \(IV^+\) or \(IV^−\)) if the discriminant \(EG - F^2\) of the first
fundamental form is positive or negative, where \(E, F, G\) are the coefficients
of the line element of \(M^2\).
In the rest of this paper we distinguish two cases according to whether these surfaces are given by (1.2a) or (1.2b) or (1.3).

It is well-known that if \( \phi : M^2 \to R, (u, v) \to \phi(u, v) \) is a smooth function, the Laplacian of the surface with respect to the induced metric can be written as

\[
\Delta \phi = -\frac{1}{\sqrt{EG - F^2}} \left[ \left( \frac{G \phi_u - F \phi_v}{\sqrt{EG - F^2}} \right)_u + \left( \frac{-F \phi_u + E \phi_v}{\sqrt{EG - F^2}} \right)_v \right]
\]

3. Helicoidal surfaces of type I

Let \( a(u) = (0, u, g(u)), u \in I \) be a \( C^2 \)-curve on any open interval of \( R \setminus \{0\} \). As mentioned earlier, by applying a helicoidal motion on this curve, we can get the helicoidal surface \( M^2 \) of \( R^3 \) which is described by Eq. (1.2a) or equivalently by

\[
r(u, v) = (u \sinh v, u \cosh v, g(u) + cv) \quad u \in I, \quad v \in R, \quad c > 0 \quad (2.1)
\]

These are helicoidal surfaces of type I.

We shall study the helicoidal surfaces in \( R^3 \) of type I given by (2.1) and satisfying the condition (1.1).

We have

\[
E = 1 + g'^2, \quad F = cg', \quad G = c^2 - u^2, \quad \text{and} \quad EG - F^2 = c^2 - u^2 (1 + g'^2)
\]

where the prime denotes derivative with respect to \( u \), and

\[
\Delta r(u, v) = (\Delta r_1(u, v), \Delta r_2(u, v), \Delta r_3(u, v)) \quad (2.2)
\]

where \( r_1(u, v) = u \sinh v, \quad r_2(u, v) = u \cosh v, \quad r_3(u, v) = g(u) + cv \).

3.1. Helicoidal surfaces of type \( I^+ \).

Suppose that \( EG - F^2 = c^2 - u^2 (1 + g'^2) > 0 \). The case \( EG - F^2 < 0 \) will be treated similarly and the corresponding results will be obtained. Let

\[
A_1(u) = \frac{1}{[c^2 - u^2(1 + g'^2)]^2} \left( u^3 g'^2 - 2c^2 u g^2 - c^2 u^2 g' g'' + u^4 g' g'' + u^3 g'^4 \right),
\]
\[ B_1(u) = \frac{1}{c^2 - u^2(1 + g'^2)} \left( 2c^3 g' + c^3 u g'' - cu^3 g'' - cu^2 g' - cu^2 g'^3 \right) \]

and

\[ C_1(u) = \frac{1}{c^2 - u^2(1 + g'^2)} \left( 2c^2 u g' + c^2 u^2 g'' - u^3 g' - u^3 g'^3 - u^4 g'' \right). \]

We have

\[
\begin{align*}
\Delta (u \sinh v) &= A_1(u) \sinh v + B_1(u) \cosh v \quad \text{(2.3)} \\
\Delta (u \cosh v) &= A_1(u) \cosh v + B_1(u) \sinh v \quad \text{(2.4)} \\
\Delta (g(u) + cv) &= C_1(u) \quad \text{(2.5)}
\end{align*}
\]

We observe that

\[ A_1(u) = -g'(u)C_1(u). \]

The equation (1.1) by means of (2.1), and (2.3), (2.4), (2.5) gives rise to the following system \((S_1)\) of ordinary differential equations

\[
(S_1) \begin{cases}
\Delta (u \sinh v) &= \lambda_1 u \sinh v \\
\Delta (u \cosh v) &= \lambda_2 u \cosh v \\
\Delta (g(u) + cv) &= \lambda_3 (g(u) + cv)
\end{cases}
\]

where \(\lambda_1, \lambda_2, \text{ and } \lambda_3 \in \text{Spec}(M^2)\). This means that \(M^2\) is at most of 3- type. \((S_1)\) becomes

\[
(S_1') \begin{cases}
A_1(u) \sinh v + B_1(u) \cosh v &= \lambda_1 u \sinh v \\
A_1(u) \cosh v + B_1(u) \sinh v &= \lambda_2 u \cosh v \\
C_1(u) &= \lambda_3 (g(u) + cv)
\end{cases}
\]

Therefore the problem of classifying the helicoidal surfaces \(M^2\) satisfying (1.1) and (2.1) is reduced to the integration of the system \((S_1')\) of ordinary differential equations. From Equation (2.8), we easily deduce that \(\lambda_3 = 0\).

On the other hand, if we multiply (2.6) by \(-\sinh v\) and (2.7) by \(\cosh v\) and then add up the resulting equations, we get

\[ A_1(u) = \lambda_2 u \cosh^2 v - \lambda_1 u \sinh^2 v, \]

Similarly by multiplying (2.6) by \(\cosh v\) and (2.7) by \(-\sinh v\), it follows that \(B_1(u) = \lambda_1 - \lambda_2 u \cosh v \sinh v\) then \(\lambda_1 - \lambda_2 = 0\) or \(\lambda_1 = \lambda_2\), we shall call it simply \(\lambda\). Then the system \((S_1')\) becomes

\[
(S_1'') \begin{cases}
A_1(u) &= -g'(u)C_1(u) = \lambda u \\
B_1(u) &= 0 \\
C_1(u) &= 0
\end{cases}
\]
First case: If \( g'(u) = 0 \). Then \( \lambda = 0 \) and \( g(u) = c_1 \) with \( c_1 \in \mathbb{R} \), is the solution of the system \((S_1')\).

Therefore, the helicoidal surface is given by

\[
r(u, v) = (u \sinh v, u \cosh v, c_1 + cv) \quad c_1 \in \mathbb{R}, \ c > 0.
\]

It’s the right helicoidal in \( \mathbb{R}^3 \).

We recover the right helicoidal of type I as a harmonic surface in \( \mathbb{R}^3 \) (because \( \Delta r(u, v) = 0 \)).

Second case: If \( g'(u) \neq 0 \), then the system \((S_1')\) is equivalently reduced to a system of two equations

\[
\begin{align*}
B_1(u) &= 0 \\
C_1(u) &= 0
\end{align*}
\]

or

\[
\begin{align*}
2c^3g' + c^3ug'' - cu^2g'' - cu^2g^3 &= 0 \tag{2.9} \\
2c^2ug' + c^2u^2g'' - u^3g'' - u^3g'^3 - u^4g'' &= 0 \tag{2.10}
\end{align*}
\]

We divide the equation (2.9) by \( c \) (\( c > 0 \)) and (2.10) by \( u \) (\( u \neq 0 \)) the last system is reduced to only one ordinary differential equation

\[
(u^2 - 2c^2) g' + (u^3 - c^2u) g'' + u^2g'^3 = 0 \tag{2.11}
\]

The equation (2.11) is equivalent to the mean curvature \( H \) of \( M^2 \) is zero which means that the surfaces are minimal.

\[
H \text{ is given by } H(u) = \frac{GL + EN - 2FM}{2(EG - F^2)} \quad \text{with}
\]

\[
L = -\frac{u g''}{w}, \quad M = \frac{c}{w}, \quad N = \frac{u^2 g'}{w}, \quad w = \left[ \sqrt[3]{c^2 - u^2 (1 + g'^2)} \right]^{\frac{1}{2}}
\]

Here \( c^2 - u^2 (1 + g'^2) > 0 \) and hence

\[
H(u) = \frac{(u^2 - 2c^2) g' + (u^3 - c^2u) g'' + u^2g'^3}{2[c^2 - u^2 (1 + g'^2)]^{\frac{1}{2}}}
\]
Set \( g' = R_1 \) in the equation (2.11). We then get a Bernoulli’s equation of the form

\[
(\text{Eq}_1) \quad (u^2 - 2c^2)R_1 + (u^3 - c^2 u) R'_1 + u^2 R'^2_1 = 0
\]

By multiplying \((\text{Eq}_1)\) by \(R_1^{-3}\) we get

\[
(u^2 - 2c^2)R_1^{-2} + (u^3 - c^2 u) R'_1 R_1^{-3} = -u^2
\]

Let \( T_1 = R_1^{-2} \). So we obtain

\[
(\text{Eq}'_1) \quad (u^2 - 2c^2) T_1 - \frac{1}{2} (u - c^2 u) T'_1 = -u^2.
\]

We first give the solution of the equation without second member. One solution is

\[
T_1(u) = \frac{\alpha u^4}{|u^2 - c^2|}, \quad \alpha > 0 \quad (2.12)
\]

But from the assumption \( c^2 - u^2 (1 + g'^2) > 0 \), it follows that \( c^2 - u^2 > 0 \) and then

\[
T_1(u) = \frac{\alpha u^4}{c^2 - u^2}, \quad \alpha > 0
\]

To integrate Equation \((\text{Eq}'_1)\) we put \( \alpha = \alpha(u) \) and \( T_1 = T_1(u), \quad T'_1 = T'_1(u) \) in \((\text{Eq}'_1)\). We find

\[
\alpha(u) = \frac{1}{u^2} + \beta
\]

where

\[
\beta > -\frac{1}{c^2} \quad (\text{then we obtain} \quad \alpha(u) > 0, \quad \forall \quad u^2 < c^2) \quad \text{and then}
\]

\[
T_1(u) = \frac{u^2 + \beta u^4}{c^2 - u^2} \quad \forall \quad 0 < |u| < c \quad \text{where} \quad \beta > -\frac{1}{c^2}
\]

but \( T_1 = R_1^{-2} \) with \( R_1 = g' \), so \( g'(u) = \pm \frac{1}{u} \sqrt{\frac{c^2 - u^2}{1 + \beta u^2}} \).
Therefore \( g(u) = a_1 \pm \int \frac{1}{|u|} \sqrt{\frac{c^2 - u^2}{1 + \beta u^2}} \, du \) for all \( u \) such that \( 0 < |u| < c \)

where \( \beta > -\frac{1}{c^2} \), \( a_1 \in \mathbb{R} \) is the constant of integration.

For example, if we take \( \beta = 0 \) and \( 0 < |u| < c \), we obtain
\[
g(u) = \pm \left[ \frac{u}{|u|} \sqrt{c^2 - u^2} - c \log \left( c + \frac{u}{|u|} \sqrt{c^2 - u^2} \right) + c \log |u| \right] + a_2
\]
where \( a_2 \in \mathbb{R} \).

In this case, the helicoidal surfaces are given by

\[
r(u, v) = \left( u \sinh v, u \cosh v, a_1 \pm \int \frac{1}{|u|} \sqrt{\frac{c^2 - u^2}{1 + \beta u^2}} \, du + cv \right)
\]

for all \( u \) such that \( 0 < |u| < c \) where \( \beta > -\frac{1}{c^2} \), \( a_1 \in \mathbb{R} \).

For \( \beta = 0 \) we have \( r(u, v) = (u \sinh v, u \cosh v, g(u) + cv) \) with

\[
g(u) = \pm \left[ \frac{u}{|u|} \sqrt{c^2 - u^2} - c \log \left( c + \frac{u}{|u|} \sqrt{c^2 - u^2} \right) + c \log |u| \right] + a_2
\]

where \( 0 < |u| < c \), and \( a_2 \in \mathbb{R} \).

So we have proved

**Proposition 1.1.** Let \( r : M^2 \rightarrow \mathbb{R}^3 \) be an isometric immersion given by \( r(u, v) = (u \sinh v, u \cosh v, g(u) + cv) \) \( v \in \mathbb{R}, \ c > 0 \), \( u \in I \subset \mathbb{R} \setminus \{0\} \) where \( EG - F^2 > 0 \).

Then \( \Delta r_i = \lambda_i r_i \), \( \lambda_i \in \mathbb{R} \), if and only if the surface \( M^2 \) is minimal.

More precisely, we have

1. \( r(u, v) = (u \sinh v, u \cosh v, c_1 + cv) \) \( c_1 \in \mathbb{R}, \ c > 0 \), \( v \in \mathbb{R}, \ u \in I \) the right helicoidal of type \( I^+ \) or
2. \( r(u, v) = \left( u \sinh v, u \cosh v, a_1 \pm \int \frac{1}{|u|} \sqrt{\frac{c^2 - u^2}{1 + \beta u^2}} \, du + cv \right) \)
\[ \forall 0 < |u| < c \quad \text{where} \quad \beta > -\frac{1}{c^2}, \quad a_1 \in \mathbb{R}. \]

**Example.** For \( \beta = 0 \), we get
\[
 r(u, v) = (u \sinh v, u \cosh v, g(u) + cv)
\]
with
\[
g(u) = \pm \left[ \frac{u}{|u|} \sqrt{c^2 - u^2} - c \log \left( c + \frac{u}{|u|} \sqrt{c^2 - u^2} \right) + c \log |u| \right] + a_2
\]
or equivalently
\[
g(u) = a_2 \pm \left[ \frac{u}{|u|} \sqrt{c^2 - u^2} - c \log \left( \frac{c}{|u|} + \frac{1}{u} \sqrt{c^2 - u^2} \right) \right]
\]

\[ \forall \quad 0 < |u| < c \quad \text{, where} \quad a_2 \in \mathbb{R} \]

### 3.2. Helicoidal surfaces of type \( I^- \)

Suppose now, that \( EG - F^2 = c^2 - u^2(1 + g'^2) < 0 \), we get
\[
\begin{align*}
\Delta (u \sinh v) &= -A_1(u) \sinh v - B_1(u) \cosh v \\
\Delta (u \cosh v) &= -A_1(u) \cosh v - B_1(u) \sinh v \\
\Delta (g(u) + cv) &= -C_1(u)
\end{align*}
\]

where \( A_1(u), B_1(u), \) and \( C_1(u) \) are the functions used in the system \((S_1')\) with the condition \( EG - F^2 > 0 \).

As we did in the other case, the problem is reduced to the integration of the system \((S)\) of ordinary differential equations
\[
(S) \begin{cases} 
- (A_1(u) \sinh v + B_1(u) \cosh v) = \lambda_1 u \sinh v \\
- (A_1(u) \cosh v + B_1(u) \sinh v) = \lambda_2 u \cosh v \\
- C_1(u) = \lambda_3 (g(u) + cv)
\end{cases}
\]
or
\[
(S') \begin{cases} 
A_1(u) \sinh v + B_1(u) \cosh v = (-\lambda_1) u \sinh v \\
A_1(u) \cosh v + B_1(u) \sinh v = (-\lambda_2) u \cosh v \\
C_1(u) = (-\lambda_3) (g(u) + cv)
\end{cases}
\]
The system \((S')\) is equivalent to \((S_1')\) with \((-\lambda_i)\) instead of \(\lambda_i\). So, we get the same results as (2.12) but from the assumption \(c^2 - u^2 (1 + g'^2) < 0\), it follows that either \(c^2 - u^2 > 0\) or \(c^2 - u^2 < 0\).

If \(c^2 - u^2 > 0\), we conclude as in the proposition 1.1.

If \(c^2 - u^2 < 0\), we get

\[
T_1(u) = \frac{\alpha(u)}{u^2 - c^2} \quad \text{where} \quad \alpha(u) = -1 + \delta \quad \text{with} \quad \delta > \frac{1}{c^2}
\]

(then we obtain \(\alpha(u) > 0\), \(\forall\ u > c^2\)) and then

\[
T_1(u) = \frac{-u^2 + \delta u^4}{u^2 - c^2} \quad \forall\ |u| > c \quad \text{where} \quad \delta > \frac{1}{c^2} \quad \text{and hence}
\]

\[
g'(u) = \pm \frac{1}{|u|} \sqrt{\frac{u^2 - c^2}{\delta u^2 - 1}} \quad \forall\ |u| > c \quad \text{where} \quad \delta > \frac{1}{c^2}.
\]

Therefore,

\[
g(u) = a_2 \pm \int \frac{1}{|u|} \sqrt{\frac{u^2 - c^2}{\delta u^2 - 1}} \, du \quad \forall\ |u| > c
\]

where \(\delta > \frac{1}{c^2}\), \(a_2 \in R\)

So we have showed the following

**Proposition 1.2.** Let \(r : M^2 \longrightarrow \mathbb{R}^3\) be an isometric immersion given by

\[
r(u, v) = (u \sinh v, u \cosh v, g(u) + cv) \quad v \in R, \quad c > 0,
\]

\(u \in I \subset R \setminus \{0\}\), where \(EG - F^2 < 0\).

Then \(\Delta r_i = \lambda_i r_i\), \(\lambda_i \in R\) if and only if the surface \(M^2\) is minimal.

More precisely, we have

1. \(r(u, v) = (u \sinh v, u \cosh v, c_1 + cv) \quad c_1 \in R, \quad c > 0, \quad v \in R, \quad u \in I\)

   the right helicoidal of type I−; or

2. \(r(u, v) = \left(u \sinh v, u \cosh v, a_1 \pm \int \frac{1}{|u|} \sqrt{\frac{c^2 - u^2}{1 + \beta u^2}} \, du + cv\right)\)
Lorentz-Minkowski space

for \( 0 < |u| < c \) where \( \beta > -\frac{1}{c^2} \), \( a_1 \in \mathbb{R} \); or

3.

\[
    r(u, v) = \left( u \sinh v, u \cosh v, a_2 \pm \int \frac{1}{|u|} \sqrt{\frac{u^2 - c^2}{\delta u^2 - 1}} \, du + cv \right)
\]

for \( |u| > c \) where \( \delta > \frac{1}{c^2} \), \( a_2 \in \mathbb{R} \).

4. Helicoidal surfaces of type II

The case where \( a(u) = (u, 0, g(u)) \) for which the corresponding helicoidal surfaces of type \( II^+ \) (or \( II^- \)) are given by

\[
    r(u, v) = (f(u) \cosh v, f(u) \sinh v, g(u) + cv) \quad f(u) > 0, \ c > 0,
\]
or equivalently by

\[
    r(u, v) = (u \cosh v, u \sinh v, g(u) + cv)
\]
is really similar to the previous one, henceforth we omit it.

5. Helicoidal surfaces of type III

Let \( a(u) = (g(u), f(u), 0), \ u \in I \) be a \( C^2 \)–curve in \( \mathbb{R}^3 \) where \( I \) is any open interval of \( \mathbb{R} \setminus \{0\} \). By applying a helicoidal motion on it, with axis \( Ox \) (timelike axis), we get the helicoidal surface \( M^2 \) in \( \mathbb{R}^3 \) given by the equation

\[
     r(u, v) = (g(u) + cv, u \cos v, u \sin v) \quad 0 \leq v \leq 2\pi, \ c > 0 \quad (3.1)
\]

where we have assumed that \( f(u) = u \).

These are helicoidal surfaces of type \( III \).

Let us study the helicoidal surfaces in \( \mathbb{R}^3 \) of type \( III \) given by (3.1) satisfying the condition (1.1).

We have \( E = 1 - g'^2(u) \), \( F = -cg'(u) \), \( G = u^2 - c^2 \) and \( EG - F^2 = u^2 (1 - g'^2) - c^2 \) where the prime denotes derivative with respect to \( u \).

5.1. Helicoidal surfaces of type \( III^+ \).

Assume that \( EG - F^2 = u^2 (1 - g'^2) - c^2 > 0 \). We get
\[ \begin{align*}
\Delta (g(u) + cv) &= C_2(u) \\
\Delta (u \cos v) &= A_2(u) \cos v + B_2(u) \sin v \\
\Delta (u \sin v) &= A_2(u) \sin v - B_2(u)
\end{align*} \]

where

\[
A_2(u) = \frac{1}{[u^2 (1 - g'^2) - c^2]^2} \left( 2 c^2 u g'^2 + c^2 u^2 g'g'' + u^3 g'^4 - u^3 g'^2 - u^4 g'g'' \right),
\]

\[
B_2(u) = \frac{1}{[u^2 (1 - g'^2) - c^2]^2} \left( c u^2 g' + c u^3 g'' - 2 c^3 g' - c^2 u^2 g'' - c^3 u g'' \right),
\]

and

\[
C_2(u) = \frac{1}{[u^2 (1 - g'^2) - c^2]^2} \left( 2 c^2 u g' + c^2 u^2 g'' + u^3 g'^3 - u^3 g' - u^4 g'' \right).
\]

We observe that

\[
A_2(u) = g'(u) \ C_2(u)
\]

The equation (1.1) by means of (3.1) and (3.2), (3.3), (3.4) gives rise to the following system \((S_2)\) of ordinary differential equations

\[\begin{align*}
\Delta (u \cos v) &= \lambda_1 u \cos v \\
\Delta (u \sin v) &= \lambda_2 u \sin v \\
\Delta (g(u) + cv) &= \lambda_3 \ (g(u) + cv)
\end{align*}\]

which becomes

\[\begin{align*}
A_2(u) \cos v + B_2(u) \sin v &= \lambda_1 u \cos v \\
A_2(u) \sin v - B_2(u) \cos v &= \lambda_2 u \sin v \\
C_2(u) &= \lambda_3 \ (g(u) + cv)
\end{align*}\]

Therefore the problem of classifying the helicoidal surfaces \(M^2\) satisfying (1.1) and (3.1) is reduced to the integration of the system \((S_2')\) of ordinary differential equations.

From the equation (3.8), we easily deduce that \(\lambda_3 = 0.0\). On the other hand, if we multiply (3.6) by \(\cos v\) and (3.7) by \(\sin v\), and then add the resulting, we get

\[
A_2(u) = \lambda_1 u \cos^2 v + \lambda_2 u \sin^2 v
\]
Similarly by (3.6) sin \( v \) + (3.7) (− cos \( v \)) it follows that
\[ B_2(u) = (\lambda_1 - \lambda_2) \ u \cos v \sin v. \]

Then \( \lambda_1 = \lambda_2 = \lambda \). Hence we obtain the system:
\[
\begin{cases}
A_2(u) = g'(u)C_2(u) = \lambda \ u \\
B_2(u) = 0 \\
C_2(u) = 0
\end{cases}
\] (\( S_2 \)")

**First case**: If \( g'(u) = 0 \). Then \( \lambda = 0 \) and \( g(u) = d_1 \) with \( d_1 \in R \) is the solution of the system (\( S_2 \")).

Therefore the helicoidal surface is given by
\[ r(u, v) = (d_1 + cv, \ u \cos v, \ u \sin v) \quad d_1 \in R, \quad c > 0 \]
it’s a right helicoidal in \( R^3 \).

**Second case**: If \( g'(u) \neq 0 \). Then the system (\( S_2 \") is reduced to a system of two equations
\[
\begin{cases}
B_2(u) = 0 \\
C_2(u) = 0
\end{cases}
\]
or
\[
\begin{align*}
-2c^3g' - c^3ug'' + cu^2g'' + cu^2g' - cu^2g'^3 &= 0 \quad (3.9) \\
2c^2ug' + c^2u^2g'' - u^3g' + u^3g'^3 - u^4g'' &= 0 \quad (3.10)
\end{align*}
\]

We divide Equation (3.9) by \( c \) \( (c > 0) \) and (3.10) by \( u \) \( (u \neq 0) \), the last system reduces to only one ordinary differential equation
\[
(2c^2 - u^2)g' - (u^3 - c^2u)g'' + u^2g'^3 = 0 \quad (3.11)
\]
Here \( H(u) = \frac{(2c^2 - u^2)g' + (u^3 - c^2u)g'' - u^2g'^3}{2[u^2(1 - g'^2) - c^2]^2} \).

Hence Equation (3.11) is equivalent to *the mean curvature* \( H \) of \( M^2 \) is zero. This means that the surfaces are minimal.

Set \( g' = R_2 \) in the equation (3.11). We also get here a Bernoulli’s equation of the form
\[
(\text{Eq}_2) \quad (u^2 - 2c^2)R_2 + (u^3 - c^2u)R_2' - u^2R_2^3 = 0.
\]

By multiplying (\( \text{Eq}_2 \)) by \( R_2^{-3} \), we get
\[
(u^2 - 2c^2)R_2^{-2} + (u^3 - c^2u)R_2' R_2^{-3} = u^2
\]
Let \( T_2 = R_2^{-2} \), then we obtain
\[
(Eq'_2) \quad (u^2 - 2c^2) T_2 - \frac{1}{2} (u^3 - c^2 u) T'_2 = u^2.
\]
Recall that \( EG - F^2 = u^2 (1 - g'^2) - c^2 > 0 \), then \( c^2 - u^2 < 0 \) and then the solutions of \((Eq'_2)\) are of the form
\[
T_2(u) = \frac{\alpha(u) u^4}{u^2 - c^2} \quad \text{where} \quad \alpha(u) = \frac{1}{u^2} + \beta \quad \text{with} \quad \beta \in R^+
\]
(then we obtain \( \alpha(u) > 0 \), \( \forall \ |u| > c \))

So
\[
T_2(u) = \frac{u^2 + \beta u^4}{u^2 - c^2} \quad \forall \ |u| > c \quad \text{where} \quad \beta \in R^+ \quad \text{and then}
\]

\[
g'(u) = \pm \frac{1}{|u|} \sqrt{\frac{u^2 - c^2}{1 + \beta u^2}}
\]

Therefore,
\[
g(u) = a_3 \pm \int \frac{1}{|u|} \sqrt{\frac{u^2 - c^2}{1 + \beta u^2}} du \quad \forall \ |u| > c \quad \text{where}
\]
\[
\beta \in R^+ \quad \text{and} \quad a_3 \in R.
\]

For instance, if we get \( \beta = 0 \) and \( \forall |u| > c \), we obtain
\[
g(u) = \pm \left( -c \ \text{Arc} \cos \frac{c}{u} + \frac{u}{|u|} \sqrt{u^2 - c^2} \right) + a_4, \quad a_4 \in R.
\]

So we have proved

**Proposition 2.1.** Let \( r : M^2 \rightarrow R^3_1 \) be an isometric immersion given by
\[
r(u, v) = (g(u) + cv, u \ \cos \ v, u \ \sin \ v) \quad 0 \leq v \leq 2\pi, \ c > 0
\]
if \( u \in I \subset R \setminus \{0\} \) where \( EG - F^2 > 0 \).

Then \( \Delta r_i = \lambda_i r_i \), \( \lambda_i \in R \) if and only if the surface \( M^2 \) is minimal.

More precisely, we have
1. \( r(u, v) = (d_1 + cv, u \ \cos \ v, u \ \sin \ v) \quad d_1 \in R, \ c > 0, \ 0 \leq v \leq 2\pi, \ u \in I \)
the right helicoidal of type III⁺; or

2.

\[ r(u, v) = \left( a_3 \pm \int \frac{1}{|u|} \sqrt{\frac{u^2 - c^2}{1 + \beta u^2}} \, du + cv, u \cos v, u \sin v \right) \forall |u| > c, \]

where \( \beta \in \mathbb{R}^+ \), and \( a_3 \in \mathbb{R} \).

Example. For \( \beta = 0 \), we get

\[ r(u, v) = \left( a_4 \pm \left(-c \arccos \frac{c}{u} + \frac{u}{|u|} \sqrt{u^2 - c^2} \right) + cv, u \cos v, u \sin v \right), \]

where \( |u| > c \), \( a_4 \in \mathbb{R} \).

5.2. Helicoidal surfaces of type III⁻.

Let us now examine the case where \( EG - F^2 = u^2(1 - g^2) - c^2 < 0 \), we get

\[ \Delta (g(u) + cv) = -C_2(u) \]
\[ \Delta (u \cos v) = -A_2(u) \cos v - B_2(u) \sin v \]
\[ \Delta (u \sin v) = -A_2(u) \sin v + B_2(u) \cos v \]

where \( A_2(u), B_2(u), \) and \( C_2(u) \) are the functions used in the system \((S_2')\) with \((-\lambda_i)\) instead of \(\lambda_i\). Here we distinguish two cases because from our hypothesis one of the following two cases occur \( u^2 - c^2 > 0 \) or \( u^2 - c^2 < 0 \).

If \( u^2 - c^2 > 0 \), we conclude as in the proposition 2.1.

If \( u^2 - c^2 < 0 \), we get

\[ T_2(u) = \frac{\alpha(u) u^4}{c^2 - u^2} \quad \text{where} \quad \alpha(u) = -\frac{1}{u^2} + \rho \quad \text{with the condition} \quad \alpha(u) > 0. \]
If $\rho > \frac{1}{c^2}$ there exists $c_0 \in ]0, c[$ such that $\alpha(c_0) = 0$, then it follows that $\alpha(u) > 0$ if and only if $c_0 < |u| < c$.

Therefore

$$T_2(u) = \frac{-u^2 + \rho u^4}{c^2 - u^2} \quad \forall c_0 < |u| < c \text{ where } \rho > \frac{1}{c^2} \text{ and hence}$$

$$g'(u) = \pm \frac{1}{|u|} \sqrt{\frac{c^2 - u^2}{\rho u^2 - 1}} \quad \forall c_0 < |u| < c \text{ where } \rho > \frac{1}{c^2}$$

$$g(u) = a_5 \pm \int \frac{1}{|u|} \sqrt{\frac{c^2 - u^2}{\rho u^2 - 1}} \, du \quad \forall c_0 < |u| < c \text{ where } \rho > \frac{1}{c^2}$$

where $a_5 \in \mathbb{R}$.

Then we have proved the following

**Proposition 2.2.** Let $r : M^2 \rightarrow R_3$ be an isometric immersion given by

$$r(u, v) = (g(u) + cv, u \cos v, u \sin v) \quad 0 \leq v \leq 2\pi, \quad c > 0,$$

$u \in I \subset R \setminus \{0\}$, where $EG - F^2 < 0$.

Then $\Delta r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$ if and only if the surface $M^2$ is minimal.

More precisely, we have

1. $r(u, v) = (d_1 + cv, u \cos v, u \sin v) \quad d_1 \in \mathbb{R}, \quad c > 0, \quad 0 \leq v \leq 2\pi,$

$u \in I$

the right helicoidal of type III$^-$ or

2. $r(u, v) = \left(a_3 \pm \int \frac{1}{|u|} \sqrt{\frac{u^2 - c^2}{1 + \beta u^2}} \, du + cv, u \cos v, u \sin v \right) \quad \forall \, |u| > c$

where $\beta \in \mathbb{R}^+$, and $a_3 \in \mathbb{R}$. or

3. $r(u, v) = \left(a_5 \pm \int \frac{1}{|u|} \sqrt{\frac{c^2 - u^2}{\rho u^2 - 1}} \, du + cv, u \cos v, u \sin v \right) \quad \forall \, |u| \in ]c_0, c[$
whith \( c_0 \) a constant, \( c_0 \in ]0, c[, \ a_5 \in R \), where \( \rho > \frac{1}{c^2} \).

References

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