Chromaticity of Bipartite Graphs

with Five or Six Edges Deleted

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Abstract

For integers \( p, q, s \) with \( p \geq q \geq 2 \) and \( s \geq 0 \), let \( \mathcal{K}^{-s}_{2}(p, q) \) denote the set of 2-connected bipartite graphs which can be obtained from \( K_{p,q} \) by deleting a set of \( s \) edges. F.M.Dong et al. (Discrete Math. vol.224 (2000) 107–124) proved that for any graph \( G \in \mathcal{K}^{-s}_{2}(p, q) \) with \( p \geq q \geq 3 \) and \( 0 \leq s \leq \min\{4, q-1\} \), then \( G \) is chromatically unique. In this paper, we shall extend this result to \( p \geq q \geq 6 \) and \( 5 \leq s \leq \min\{6, q-1\} \).

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1 Introduction

All graphs considered here are simple graphs. For a graph $G$, let $V(G)$, $\Delta(G)$ and $P(G, \lambda)$ be the vertex set, maximum degree and the chromatic polynomial of $G$, respectively.

Two graphs $G$ and $H$ are said to be chromatically equivalent (or simply $\chi$–equivalent), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by $G$ under $\sim$ is denoted by $[G]$. A graph $G$ is chromatically unique (or simply $\chi$–unique) if $H \sim G$ whenever $H \sim G$, i.e., $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi$–closed.

For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1, \lambda) \neq P(G_2, \lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be chromatically disjoint, or simply $\chi$–disjoint.

For integers $p, q, s$ with $p \geq q \geq 2$ and $s \geq 0$, let $K^{-s}(p, q)$ (resp. $K^{-s}_{2}(p, q)$) denote the set of connected (resp. 2–connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of $s$ edges.

In [5], Dong et al. proved the following result.

**Theorem 1.1** For integers $p, q, s$ with $p \geq q \geq 2$ and $0 \leq s \leq q - 1$, $K^{-s}_{2}(p, q)$ is $\chi$–closed.

Teo and Koh [14] showed that every graph in $\mathcal{K}(p, q) \cup \mathcal{K}^{-1}(p, q)$ is $\chi$–unique. The case when $s \geq 2$ has been studied by Giudici and Lima de Sa [6], Peng [7], Borowiecki and Drgas-Burchardt [1]. Their typical results are of the following:

(i) If $2 \leq s \leq 4$ and $p - q$ is small enough, then each graph in $\mathcal{K}^{-s}(p, q)$ is $\chi$–unique;

(ii) If $G \in \mathcal{K}^{-s}(p, q)$, where $0 \leq p - q \leq 1$, such that the set of $s$ edges deleted forms a matching, then $G$ is $\chi$–unique.

Chen [2] showed that if $G \in \mathcal{K}^{-s}(p, q)$, where $3 \leq s \leq p - q$ and

$$q \geq \max \left\{ \frac{1}{2}(p - q)(s - 1) + \frac{3}{2}, \frac{8}{27}(p - q)^2 + \frac{1}{3}(p - q) + 5s + 6 \right\},$$

then $G$ is $\chi$–unique.
Chromaticity of bipartite graphs

and the set of $s$ edges deleted forms a matching or a star, then $G$ is $\chi$–unique. In [5], Dong et al. proved that any $2$–connected graph obtained from $K_{p,q}$ by deleting a set of edges that forms a matching of size at most $q - 1$ or that induces a star is chromatically unique.

Very recently, Dong et al. [4] showed that any graph in $K^{-s}_{2}(p,q)$ is $\chi$–unique if $p \geq q \geq 3$ and $1 \leq s \leq \min\{4, q - 1\}$. In this paper, we shall extend this result by examining the chromatic uniqueness of every $G \in K^{-s}_{2}(p,q)$, where $p \geq q \geq 6$ and $5 \leq s \leq \min\{6, q - 1\}$.

2 Preliminary results and notation

Throughout this paper, we fix the following conditions for $p, q$ and $s$:

$$p \geq q \geq 3 \quad \text{and} \quad 1 \leq s \leq q - 1.$$ 

For a bipartite graph $G = (A,B;E)$ with bipartition $A$ and $B$ and edge set $E$, let $G' = (A',B';E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E, x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$. Let $\triangle(G')$ denote the maximum degree of $G'$. Partition $K_{2}^{-s}(p,q)$ into the following subsets:

$$D_i(p,q,s) = \left\{ G \in K_{2}^{-s}(p,q) \mid \triangle(G') = i \right\}, \quad i = 1, 2, \ldots, s.$$ 

The following two results were obtained in [3].

**Theorem 2.1** Let $p, q, s$ be integers with $p \geq q \geq 3$ and $1 \leq s \leq q - 1$. The following sets are pairwise $\chi$-disjoint:

$$D_1(p,q,s), \cup_{2 \leq i \leq t} D_i(p,q,s), D_t(p,q,s), D_{t+1}(p,q,s), \ldots, D_s(p,q,s),$$

where $t = \lceil (s + 3)/2 \rceil$.

**Theorem 2.2** Let $p \geq q \geq 3$ and $1 \leq s \leq q - 1$.

(i) $D_1(p,q,s)$ is $\chi$-closed.

(ii) $\cup_{2 \leq i \leq (s+3)/2} D_i(p,q,s)$ is $\chi$-closed for $s \geq 2$. 
(iii) \( D_i(p, q, s) \) is \( \chi \)-closed for each \( i \) with \( \lceil (s + 3)/2 \rceil \leq i \leq \min\{s, q - 2\} \).

(iv) \( D_{q-1}(p, q, s) \cap K^{-s}_{2}(p, q) \) is \( \chi \)-closed for \( s = q - 1 \).

For a graph \( G \) and a positive integer \( k \), a partition \( \{A_1, A_2, \ldots, A_k\} \) of \( V(G) \) is called a \( k \)-independent partition in \( G \) if each \( A_i \) is a non-empty independent set of \( G \). Let \( \alpha(G, k) \) denote the number of \( k \)-independent partitions in \( G \). For any graph \( G \) of order \( n \), we have (see [8]):

\[
P(G, \lambda) = \sum_{k=1}^{n} \alpha(G, k)\lambda(\lambda - 1) \cdots (\lambda - k + 1).
\]

Thus, we have

**Lemma 2.1** If \( G \sim H \), then \( \alpha(G, k) = \alpha(H, k) \) for \( k = 1, 2, \ldots \).

For any bipartite graph \( G = (A, B; E) \) with bipartition \( A \) and \( B \) and edge set \( E \), let

\[
a'(G, 3) = \alpha(G, 3) - (2^{|A|-1} + 2^{|B|-1} - 2).
\]  

(1)

In [5], the authors found the following sharp bounds for \( \alpha'(G, 3) \):

**Theorem 2.3** For \( G \in K^{-s}_{2}(p, q) \) with \( p \geq q \geq 3 \) and \( 0 \leq s \leq q - 1 \),

\[
s \leq \alpha'(G, 3) \leq 2^s - 1,
\]

where \( \alpha'(G, 3) = s \) iff \( \Delta(G') = 1 \) and \( \alpha'(G, 3) = 2^s - 1 \) iff \( \Delta(G') = s \).

For \( t = 0, 1, 2, \ldots \), let \( B(p, q, s, t) \) denote the set of graphs \( G \in K^{-s}_{2}(p, q) \) with \( \alpha'(G, 3) = s + t \). Thus, \( K^{-s}_{2}(p, q) \) is partitioned into the following subsets:

\[B(p, q, s, 0), B(p, q, s, 1), \ldots, B(p, q, s, 2^s - s - 1).\]

Assume that \( B(p, q, s, t) = \emptyset \) for \( t > 2^s - s - 1 \).

**Lemma 2.2** (Dong et al. [4]) For \( p \geq q \geq 3 \) and \( 0 \leq s \leq q - 1 \), if \( 0 \leq t \leq 2^{s-1} - q - 1 \), then

\[B(p, q, s, t) \subseteq K^{-s}_{2}(p, q).\]
Dong et al. [5] have shown that any graph \( G \) in \( B(p, q, s, 0) \cup B(p, q, s, 2^s-s-1) \), if \( G \) is 2-connected, is \( \chi \)-unique. In [4], Dong et al. proved that every 2-connected graph in \( B(p, q, s, t) \) is \( \chi \)-unique for \( 1 \leq t \leq 4 \). Roslan and Peng in [10,11,12] proved that every 2-connected graph in \( B(p, q, s, t) \) is \( \chi \)-unique for \( 5 \leq t \leq 7 \).

For a bipartite graph \( G = (A, B; E) \), let

\[
\Omega(G) = \{ Q \mid Q \text{ is an independent sets in } G \text{ with } Q \cap A \neq \emptyset, Q \cap B \neq \emptyset \}.
\]

**Lemma 2.3** (Dong et al. [5]) For \( G \in K^{-s}(p, q) \),

\[
\alpha'(G, 3) = |\Omega(G)| \geq 2^{\Delta(G')} + s - 1 - \Delta(G').
\]

For a bipartite graph \( G = (A, B; E) \), the number of 4-independent partitions \( \{A_1, A_2, A_3, A_4\} \) in \( G \) with \( A_i \subseteq A \) or \( A_i \subseteq B \) for all \( i = 1, 2, 3, 4 \) is

\[
(2^{|A|-1} - 1)(2^{|B|-1} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3)
\]

\[
= (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2.
\]

Define

\[
\alpha'(G, 4) = \alpha(G, 4) - \{ (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2 \}.
\]

Observe that for \( G, H \in K^{-s}(p, q) \),

\[
\alpha(G, 4) = \alpha(H, 4) \iff \alpha'(G, 4) = \alpha'(H, 4).
\]

Note that by Lemma 2.1, we introduced \( \alpha'(G, 3) \) and \( \alpha'(G, 4) \) to study the chromaticity of graphs \( G \in K^{-s}(p, q) \). The following results will be used to prove our main theorems.

**Lemma 2.4** (Dong et al. [3]) For \( G = (A, B; E) \in K^{-s}(p, q) \) with \(|A| = p \) and \(|B| = q\),

\[
\alpha'(G, 4) = \sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2) + \left| \{ \{Q_1, Q_2\} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset \} \right|.
\]
Lemma 2.5 (Dong et al. [4]) For a bipartite graph \( G = (A, B; E) \), if \( uvw \) is a path in \( G' \) with \( d_{G'}(u) = 1 \) and \( d_{G'}(v) = 2 \), then for any \( k \geq 2 \),

\[
\alpha(G, k) = \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1). 
\]

Lemma 2.6 (Roslan and Peng [9]) For a bipartite graph \( G = (A, B; E) \), if \( uvw, uvy \) and \( wvy \) are three paths in \( G' \) with \( d_{G'}(u) = 1 \) and \( d_{G'}(v) = 3 \), then for any \( k \geq 2 \),

\[
\alpha(G, k) = \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1) + \\
\alpha(G - \{u, v, y\}, k - 1) + \alpha(G - \{u, v, w, y\}, k - 1). 
\]

Theorem 2.4 (Dong et al. [3])

(a) For any \( G \in \mathcal{K}_2^s(p, q) \), with \( p \geq q \geq s + 1 \geq 6 \), if \( \Delta(G') = s - 1 \), then \( G \) is \( \chi \)-unique.

(b) For any \( G \in \mathcal{K}_2^s(p, q) \), with \( p \geq q \geq s + 1 \geq 8 \), if \( \Delta(G') = s - 2 \), then \( G \) is \( \chi \)-unique.

Theorem 2.5

(a) For any \( G \in \mathcal{B}(p, q, s, 0) \cup \mathcal{B}(p, q, s, 2^s - s - 1) \), if \( G \) is 2-connected, then \( G \) is \( \chi \)-unique. (Dong et al. [5])

(b) For any \( G \in \bigcup_{t=1}^4 \mathcal{B}(p, q, s, t) \), if \( G \) is 2-connected, then \( G \) is \( \chi \)-unique. (Dong et al. [4])

(c) For any \( G \in \bigcup_{t=3}^7 \mathcal{B}(p, q, s, t) \), if \( G \) is 2-connected, then \( G \) is \( \chi \)-unique. (Roslan and Peng [10,11,12])

For convenient we define the graphs \( Y_n, Z_1, Z_2 \) and \( Z_3 \) as in Figure 1.
3 Main result

Dong et al. in [4] proved that every graph in $K_{2}^{-s}(p, q)$ is $\chi$-unique if $p \geq q \geq 3$ and $1 \leq s \leq \min\{4, q - 1\}$. In this section, we shall show that every graph in $K_{2}^{-s}(p, q)$ is $\chi$-unique if $p \geq q \geq 6$ and $5 \leq s \leq \min\{6, q - 1\}$. 
TABLE 1: 5 Deleted Edges

| Name of Graph, $G_i$ | Graphs $G'_i$ | $|A|=p$, $|B|=q$ | $\alpha'(G_i, 4) - 5(2^{p-2} + 2^{q-2} - 2)$ |
|---------------------|--------------|----------------|--------------------------|
| $G_1$               | ![Graph 1](image) | $A$ | $4(2^{p-3} + 2^{q-2} - 2) + 2(2^{p-3} + 2^{q-2} - 2) + (2^{p-3} + 2^{q-3} - 2) + (2^{p-4} + 2^{q-2} - 2) + 7$ |
| $G_2$               | ![Graph 2](image) | $A$ | $4(2^{p-3} + 2^{q-2} - 2) + 2(2^{p-3} + 2^{q-2} - 2) + (2^{p-3} + 2^{q-3} - 2) + (2^{p-4} + 2^{q-2} - 2) + 7$ |
| $G_3$               | ![Graph 3](image) | $A$ | $3(2^{p-3} + 2^{q-2} - 2) + 3(2^{p-3} + 2^{q-2} - 2) + (2^{p-4} + 2^{q-2} - 2) + (2^{p-4} + 2^{q-2} - 2) + 9$ |

Our main results are Theorems 3.1 and 3.2.

**Theorem 3.1** Every graph in $K_2^{-5}(p, q)$ with $p \geq q \geq 6$ is $\chi$-unique.

**Proof.** Let $G \in K_2^{-s}(p, q)$ with $p \geq q \geq 6$ and $s = 5$. If $\Delta(G') \in \{1, 5\}$, then $\alpha'(G, 3) = s$ or $\alpha'(G, 3) = 2^s - 1$ and thus $G$ is $\chi$-unique by Theorem 2.5(a). If $\Delta(G') = 4$, then $G$ is $\chi$-unique by Theorem 2.4(a). If $\Delta(G') = 2$ and $G' \not\cong K_{2,2} \cup K_2$, then $\alpha'(G, 3) \leq s + 4$ and thus $G$ is $\chi$-unique by Theorem 2.5(b). If $G' \cong K_{2,2} \cup K_2$, then $\alpha'(G, 3) = s+5$ and thus $G$ is $\chi$-unique by Theorem 2.5(c). Let $\Delta(G') = 3$. If $G' \cong K_{1,3} \cup 2K_2$, then $\alpha'(G, 3) = 9 = s + 4$ and thus $G$ is $\chi$-unique by Theorem 2.5(b); if $G' \cong K_{1,3} \cup P_3$ or $G' \cong Y_3 \cup K_2$, then $\alpha'(G, 3) = 10 = s + 5$ and thus $G$ is $\chi$-unique by Theorem 2.5(c); if $G' \cong Z_1$ or $G' \cong Y_4$, then $\alpha'(G, 3) = 11 = s + 6$ and thus $G$ is $\chi$-unique by Theorem 2.5(c);
otherwise, there are two more possible structures for \( G' \) and they are shown in Table 1. For \( i = 1, 2, 3 \), \( \alpha'(G_i, 4) \) is obtained by Lemma 2.4.

When \( p = q \), \( G_1 \cong G_2 \) and \( \alpha'(G_1, 4) - \alpha'(G_3, 4) = -4 \cdot 2^{p-4} + 5 \cdot 2^{q-4} - 2 > 0 \).

When \( p > q \), we have \( \alpha'(G_1, 4) - \alpha'(G_3, 4) = -4 \cdot 2^{p-4} + 5 \cdot 2^{q-4} - 2 < 0 \); and \( \alpha'(G_3, 4) - \alpha'(G_2, 4) = -3 \cdot 2^{p-4} + 2 \cdot 2^{q-4} + 2 < 0 \).

Therefore, \( \alpha'(G_1, 4) \neq \alpha'(G_3, 4) \neq \alpha'(G_2, 4) \). Hence, \( G \) is also \( \chi \)-unique when \( G' \) is the structure in Table 1. This completes the proof.

\textbf{Theorem 3.2} Every graph in \( K_2^{-6}(p, q) \) with \( p \geq q \geq 7 \) is \( \chi \)-unique.

\textbf{Proof.} Let \( G \in K_2^{-s}(p, q) \) with \( p \geq q \geq 7 \) and \( s = 6 \). If \( \Delta(G') \in \{1, 6\} \), then \( \alpha'(G, 3) = s \) or \( \alpha'(G, 3) = 2^s - 1 \) and thus \( G \) is \( \chi \)-unique by Theorem 2.5(a).

If \( \Delta(G') = 5 \), then \( G \) is \( \chi \)-unique by Theorem 2.4(a).

Let \( \Delta(G') = 2 \). If \( G' \not\cong C_4 \cup 2K_2 \) or \( G' \not\cong C_4 \cup P_3 \) or \( G' \not\cong C_6 \), then \( \alpha'(G, 3) \leq s + 6 \) and thus \( G \) is \( \chi \)-unique by Theorems 2.5(b) and (c); if \( G' \cong C_4 \cup 2K_2 \), then \( \alpha'(G, 3) = 11 = s + 5 \) and thus \( G \) is \( \chi \)-unique by Theorem 2.5(c); and if \( G' \cong C_4 \cup P_3 \) or \( G' \cong C_6 \), then \( \alpha'(G, 3) = 12 = s + 6 \) and thus \( G \) is \( \chi \)-unique by Theorem 2.5(c).

Let \( \Delta(G') \in \{3, 4\} \). If \( G' \cong K_{1,3} \cup 3K_2 \), then \( \alpha'(G, 3) = 10 = s + 4 \) and thus \( G \) is \( \chi \)-unique by Theorem 2.5(b); if \( G' \cong K_{1,3} \cup P_3 \cup K_2 \) or \( G' \cong Y_3 \cup 2K_2 \), then \( \alpha'(G, 3) = 11 = s + 5 \) and thus \( G \) is \( \chi \)-unique by Theorem 2.5(c); if \( G' \cong Y_4 \cup K_2 \) or \( G' \cong Y_3 \cup P_3 \) or \( G' \cong Z_1 \cup K_2 \) or \( G' \cong K_{1,3} \cup P_4 \), then \( \alpha'(G, 3) = 12 = s + 6 \) and thus \( G \) is \( \chi \)-unique by Theorem 2.5(c); if \( G' \cong Z_3 \) or \( G' \cong Z_2 \) or \( G' \cong Y_5 \), then \( \alpha'(G, 3) = 13 = s + 7 \) and thus \( G \) is \( \chi \)-unique by Theorem 2.5(c); otherwise, there are 33 more possible structures for \( G' \) and they are named as \( G_1', G_2', \ldots, G_{33}' \) (see Table 2 in [14]).

We group \( G_1, G_2, \ldots, G_{33} \) according to their values of \( \alpha'(G_i, 3) \) which can be computed by using Lemma 2.3 and these values are shown in column three of Table 2 ([14]). Let

\[
T_1 = \{ G_1, G_2, G_3, G_4 \}
\]

\[
T_2 = \{ G_5, G_6 \}
\]
\( \mathcal{T}_3 = \{ G_7, G_8, G_9, G_{10} \} \)
\( \mathcal{T}_4 = \{ G_{11}, G_{12}, G_{13}, G_{14}, G_{15}, G_{16} \} \)
\( \mathcal{T}_5 = \{ G_{17}, G_{18}, G_{19}, G_{20}, G_{21} \} \)
\( \mathcal{T}_6 = \{ G_{22}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27} \} \)
\( \mathcal{T}_7 = \{ G_{28}, G_{29}, G_{30}, G_{31}, G_{32}, G_{33} \} \)

Observe that for any \( i, j \) with \( 1 \leq i < j \leq 7 \), \( \alpha'(G, 3) > \alpha'(H, 3) \) if \( G \in \mathcal{T}_i \) and \( H \in \mathcal{T}_j \). Thus by Lemma 2.1 and Equation (1), \( \mathcal{T}_i \) and \( \mathcal{T}_j \) (\( 1 \leq i < j \leq 7 \)) are \( \chi \)-disjoint and since \( \bigcup_{i=2}^{7} \mathcal{D}_i(p, q, s) \) is \( \chi \)-closed (see Theorem 2.2), each \( \mathcal{T}_i \) (\( 1 \leq i \leq 7 \)) is \( \chi \)-closed. Hence, for each \( i \), to show that all graphs in \( \mathcal{T}_i \) are \( \chi \)-unique, it suffices to show that for any two graphs \( G, H \in \mathcal{T}_i \), if \( G \not\cong H \), then either \( \alpha'(G, 4) \neq \alpha'(H, 4) \) or \( \alpha(G, 5) \neq \alpha(H, 5) \). The remaining works is to compare every two graphs in each \( \mathcal{T}_i \) for \( 1 \leq i \leq 7 \).

We shall establish several inequalities of the form \( \alpha'(G_i, 4) < \alpha'(G_j, 4) \) for some \( i, j \). Since the method used to obtain these inequalities is standard, long and rather repetitive, we shall not discuss all here. As an example of detail comparison, we only show that \( \alpha(G_{11}, 5) - \alpha(G_{12}, 5) > 0 \) when \( p = q \) in the family \( \mathcal{T}_4 \). The reader may refer to [15] for all graphs and other detail comparisons.

(1) \( \mathcal{T}_1 \).

(1.1) When \( p = q \), \( G_1 \cong G_2, G_3 \cong G_4 \) and \( \alpha'(G_3, 4) < \alpha'(G_1, 4) \).

(1.2) When \( p > q \), \( \alpha'(G_3, 4) < \alpha'(G_1, 4) < \alpha'(G_2, 4) < \alpha'(G_4, 4) \).

(2) \( \mathcal{T}_2 \).

(2.1) When \( p = q \), \( G_5 \cong G_6 \).

(2.2) When \( p > q \), \( \alpha'(G_5, 4) < \alpha'(G_6, 4) \).

(3) \( \mathcal{T}_3 \).

(3.1) When \( p = q \), \( G_7 \cong G_8, G_9 \cong G_{10} \) and \( \alpha'(G_7, 4) < \alpha'(G_9, 4) \).

(3.2) When \( p > q \), \( \alpha'(G_9, 4) < \alpha'(G_7, 4) < \alpha'(G_8, 4) < \alpha'(G_{10}, 4) \).

(4) \( \mathcal{T}_4 \).

Note that \( \alpha'(G_i, 4) \) is odd when \( 11 \leq i \leq 14 \) and is even when \( i = 15, 16 \). Thus \( \alpha'(G_i, 4) \neq \alpha'(G_j, 4) \) if \( 11 \leq i \leq 14 \) and \( j = 15, 16 \).

(4.1) The graph \( G_i \) when \( 11 \leq i \leq 14 \).
(4.1.1) When \( p = q \), \( G_{11} \cong G_{14}, G_{12} \cong G_{13} \) and \( \alpha'(G_{11}, 4) = \alpha'(G_{12}, 4) \), and by Lemma 2.5,

\[
\alpha(G_{11}, 5) - \alpha(G_{12}, 5) = \left[ \alpha(G_{11} + a_1b_1, 5) + \alpha(G_{11} - \{a_1, b_1\}, 4) + \alpha(G_{11} - \{a_1, b_1, c_1\}, 4) \right] - \\
\left[ \alpha(G_{12} + a_2b_2, 5) + \alpha(G_{12} - \{a_2, b_2\}, 4) + \alpha(G_{12} - \{a_2, b_2, c_2\}, 4) \right]
\]

\[
= \alpha(G_{11} - \{a_1, b_1, c_1\}, 4) - \alpha(G_{12} - \{a_2, b_2, c_2\}, 4)
\]

\[
= \alpha'(G_{11} - \{a_1, b_1, c_1\}, 4) - \alpha'(G_{12} - \{a_2, b_2, c_2\}, 4)
\]

\[
= \left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{p-i-3} + 2^{q-3} - 2) \right\} - \left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{p-i-2} + 2^{q-4} - 2) \right\}
\]

\[
= \sum_{i=1}^{4} \binom{4}{i} (2^{q-4} - 2^{p-i-3}) > 0,
\]

since \( G_{11} + a_1b_1 \cong G_{12} + a_2b_2 \) and \( G_{11} - \{a_1, b_1\} \cong G_{12} - \{a_2, b_2\} \).

(4.1.2) When \( p > q \), \( \alpha'(G_{11}, 4) < \alpha'(G_{12}, 4) < \alpha'(G_{13}, 4) < \alpha'(G_{14}, 4) \).

(4.2) The graph \( G_i \) when \( i = 15, 16 \).

(4.2.1) When \( p = q \), \( G_{15} \cong G_{16} \).

(4.2.2) When \( p > q \), \( \alpha'(G_{15}, 4) < \alpha'(G_{16}, 4) \).

(5) \( \mathcal{T}_5 \).

Note that \( \alpha'(G_i, 4) \) is odd when \( 17 \leq i \leq 20 \) and even when \( i = 21 \). Thus \( \alpha'(G_i, 4) \neq \alpha'(G_{21}, 4) \) for \( 17 \leq i \leq 20 \). Hence, we only need to consider the graphs \( G_i \) when \( 17 \leq i \leq 20 \).

(5.1) When \( p = q \), \( G_{17} \cong G_{18}, G_{19} \cong G_{20} \) and \( \alpha'(G_{17}, 4) < \alpha'(G_{19}, 4) \).

(5.2) When \( p > q \), \( \alpha'(G_{17}, 4) < \alpha'(G_{19}, 4) < \alpha'(G_{20}, 4) < \alpha'(G_{18}, 4) \).

(6) \( \mathcal{T}_6 \).

(6.1) When \( p = q \), \( G_{22} \cong G_{23}, G_{24} \cong G_{25}, G_{26} \cong G_{27} \) and \( \alpha'(G_{26}, 4) < \alpha'(G_{22}, 4) < \alpha'(G_{24}, 4) \).

(6.2) When \( p > q \), if \( p - q = 1 \),

\[
\alpha'(G_{24}, 4) < \alpha'(G_{26}, 4) < \alpha'(G_{22}, 4) < \alpha'(G_{27}, 4) < \alpha'(G_{23}, 4) < \alpha'(G_{25}, 4);
\]

and if \( p - q \geq 2 \),

\[
\alpha'(G_{24}, 4) < \alpha'(G_{26}, 4) < \alpha'(G_{22}, 4) < \alpha'(G_{23}, 4) < \alpha'(G_{27}, 4) < \alpha'(G_{25}, 4).
\]
Thus $\alpha'(G_i, 4) \neq \alpha'(G_j, 4)$ for $22 \leq i < j \leq 27$.

(7) $T_7$.

Note that $\alpha'(G_i, 4)$ is odd when $i = 31, 32, 33$ and even when $i = 28, 29, 30$.

Thus $\alpha'(G_i, 4) \neq \alpha'(G_j, 4)$ for $i = 31, 32, 33$ and $j = 28, 29, 30$.

(7.1) The graphs $G_i$ when $i = 28, 29, 30$.

(7.1.1) When $p = q$, $G_{28} \cong G_{29}$ and $\alpha'(G_{28}, 4) < \alpha'(G_{30}, 4)$.

(7.1.2) When $p > q$, $\alpha'(G_{28}, 4) < \alpha'(G_{30}, 4) < \alpha'(G_{29}, 4)$.

(7.2) The graphs $G_i$ when $i = 31, 32, 33$.

(7.2.1) When $p = q$, $G_{31} \cong G_{33}$ and $\alpha'(G_{31}, 4) = \alpha'(G_{32}, 4)$, and by using Lemma 2.6, $\alpha(G_{31}, 5) > \alpha(G_{32}, 5)$.

(7.2.2) When $p > q$, $\alpha'(G_{31}, 4) < \alpha'(G_{32}, 4) < \alpha'(G_{33}, 4)$.

Hence, $G$ is also $\chi$-unique when $G'$ is one of the structures in Table 2 [14].

This completes the proof of the theorem.

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References


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