Expansion Mapping Theorems in Metric Spaces

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Abstract

In this paper, we prove expansion mappings theorems in metric spaces, which generalizes Theorem 1.3 of Kumar [Common fixed point theorems for expansion mappings in various spaces, Acta.Math. Hunger. 118(1-2) (2008),9-28] . Also we introduce the concept of R-weak commutativity of type (P) in metric spaces and provide various examples to reflect upon the distinctiveness among the R-weakly commuting mapping of type (A_g), R-weakly commuting mapping of type (A_i) and R-weakly commuting mapping of type (P). We also discuss some results related to R-weak commuting type mappings.

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In 1922, the Polish mathematician, Banach proved a common fixed-point theorem, which ensures the existence and uniqueness of a fixed-point, under appropriate conditions. This result of Banach is known as Banach's fixed point theorem or Banach contraction principle, which state that “let (X,d) be a complete
metric space. If $T$ satisfies $d(Tx, Ty) \leq kd(x, y)$ for each $x, y$ in $X$ where $0 < k < 1$, then $T$ has a unique fixed point in $X$. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Many authors have extended, generalized and improved Banach fixed point theorem in different ways. For the last quarter of the 20th century, there has been a considerable interest in the study of common fixed point of pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were obtained in this direction by various authors. The generalization of Banach’s fixed point theorem by Jungck [4] gave a new direction to the “Fixed point theory and its applications.” This theorem has had many applications, but suffers from the drawback that the definition requires that $T$ be continuous throughout $X$. There then follows a flood of papers involving contractive definition that do not require the continuity of $T$. This result was further generalized and extended in various ways by many authors. On the other hand, S. Sessa [13] coined the notion of weak commutativity and proved common fixed point theorem for these mappings.

**Definition 1.1.** Two self-mappings $f$ and $g$ be of a metric space $(X, d)$ are said to be weakly commuting if  
\[ d(fgx, gfx) \leq d(gx, fx) \text{ for all } x \text{ in } X. \]

Further, Jungck [5] introduced more generalized commutativity, so called compatibility, which is more general than that of weak commutativity.

**Definition 1.2.** Two self-mappings $f$ and $g$ be of a metric space $(X, d)$ are said to be compatible if  
\[ \lim_{n \to \infty} d(fgx_n, gfx_n) = 0, \text{ whenever } \{x_n\}_{n=1}^{\infty} \text{ is a sequence in } X \text{ such that } \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \text{ for some } t \text{ in } X. \]

This concept has been useful for obtaining fixed point theorems for compatible mappings satisfying contractive conditions and assuming continuity of at least one of the mappings. It has been known from the paper of Kannan [7] that there exists maps that have a discontinuity in the domain but which have fixed points, moreover, the maps involved in every case were continuous at the fixed point. This paper was a genesis for a multitude of fixed point papers over the next two decades.

In 1994, Pant [10] introduced the notion of R-weakly commuting mappings in metric spaces, firstly to widen the scope the study of common fixed point theorems from the class of compatible to the wider class of R-weakly commuting mappings. Secondly, maps are not necessarily continuous at the fixed point.
Definition 1.3. A pair of self-mappings \((f, g)\) of a metric space \((X, d)\) is said to be R-weakly commuting if there exists some \(R > 0\) such that
\[ d(fgx, gfx) \leq Rd(fx, gx) \]
for all \(x \in X\).

In 1997, Pathak Cho and Kang [9] introduced the improved notions of R-weakly commuting mappings and called them R-weakly commuting mappings of type \((A_f)\) and R-weakly commuting mappings of type \((A_g)\).

Definition 1.4. A pair of self-mappings \((f, g)\) of a metric space \((X, d)\) is said to be
(i) R-weakly commuting mappings of type \((A_f)\) if there exists some \(R > 0\) such that
\[ d(fgx, ggx) \leq R(d(fx, gx)) \]
for all \(x \in X\).
(ii) R-weakly commuting mappings of type \((A_g)\) if there exists some \(R > 0\) such that
\[ d(gfx, ffx) \leq Rd(fx, gx) \]
for all \(x \in X\).

In 1996, Jungck [6] introduced the concept of weakly compatible maps as follows:

Definition 1.5. Two self maps \(f\) and \(g\) are said to be weakly compatible if they commute at coincidence points.

Example 1.1. Let \(X = \mathbb{R}\) and define \(f, g : \mathbb{R} \to \mathbb{R}\) by \(fx = \frac{x}{3}, x \in \mathbb{R}\) and \(gx = x^2, x \in \mathbb{R}\). Here 0 and 1/3 are two coincidence points for the maps \(f\) and \(g\). Note that \(f\) and \(g\) commute at 0, i.e., \(fg(0) = gf(0) = 0\), but \(fg(1/3) = f(1/9) = 1/27\) and \(gf(1/3) = g(1/9) = 1/81\) and so \(f\) and \(g\) are not weakly compatible maps on \(\mathbb{R}\).

Example 1.2. Weakly compatible maps need not be compatible. Let \(X = [2, 20]\) and \(d\) be the usual metric on \(X\). Define mappings \(B, T : X \to X\) by
\[ Bx = x \text{ if } x = 2 \text{ or } x > 5, \quad Bx = 6 \text{ if } 2 < x \leq 5, \quad Tx = x \text{ if } x = 2, \quad Tx = 12 \text{ if } 2 < x \leq 5, \quad Tx = x - 3 \text{ if } x > 5. \]
The mappings \(B\) and \(T\) are non-compatible since sequence \(\{x_n\}\) defined by \(x_n = 5 + (1/n), n \geq 1\). Then \(Tx_n \to 2, Bx_n \to 2, TBx_n = 2\) and \(BTx_n = 6\). But they are weakly compatible since they commute at coincidence point at \(x = 2\).

Now we introduce the following notions which seem to be unreported.

Definition 1.6. A pair of self-mappings \((f, g)\) of a metric space \((X, d)\) is said to be R-weakly commuting mappings of type \((P)\) if there exists some \(R > 0\) such that
\[ d(ffx, ggx) \leq Rd(fx, gx) \]
for all \(x \in X\).
Remark 1.1. We have suitable examples which show that R-weakly commuting mappings, R-weakly commuting of type \((A_f)\), R-weakly commuting of type \((A_g)\) and R-weakly commuting of type \((P)\) are distinct.

Example 1.3. Let \(X = [-1,1]\) the set of all real numbers with usual metric \(d\) defined by
\[
d(x, y) = |x - y| \quad \text{for all } x, y \in X.
\]
Define \(f_x = |x|\) and \(g_x = |x| - 1\). Then by a straightforward calculation, one can show that
\[
d(f_x, g_x) = 1, \quad d(f_{gx}, g_{fx}) = 2(1 - |x|), \quad d(f_{gx}, gg_x) = 1, \quad d(g_{fx}, ff_x) = 1, \quad d(ff_x, gg_x) = 2|x|\quad \text{for all } x, y \in X.
\]
Now we conclude the following:
\(i\) pair \((f, g)\) is not weakly commuting
\(ii\) for \(R=2\), pair \((f, g)\) is R-weakly commuting, R-weakly commuting of type \((P)\), R-weakly commuting of type \((A_f)\) and R-weakly commuting of type \((A_g)\).
\(iii\) For \(R = \frac{3}{2}\), pair \((f, g)\) is R-weakly commuting of type \((A_f)\) but not R-weakly commuting of type \((P)\) and R-weakly commuting.

Example 1.4. Let \(X = [0,1]\) the set of all real numbers with usual metric \(d\) defined by
\[
d(x, y) = |x - y| \quad \text{for all } x, y \in X.
\]
Define \(f_x = x\) and \(g_x = x^2\). Then by a straightforward calculation, one can show that
\[
d(f_{gx}, gg_x) = 0, \quad d(f_{gx}, gg_x) = |x^2(x - 1)(x + 1)|, \quad d(g_{fx}, ff_x) = |x(x - 1)|, \quad d(ff_x, gg_x) = \left|x(x - 1)(x^2 + x + 1)\right| \quad \text{and} \quad d(ff_x, gg_x) = |x(x - 1)|\quad \text{for all } x, y \in X.
\]
Therefore, we conclude that
\(a\). pair \((f, g)\) is R-weakly commuting for all positive, real values of \(R\).
\(b\). For \(R=3\), pair \((f, g)\) is R-weakly commuting of the type \((A_f)\), R-weakly commuting of the type \((A_g)\) and R-weakly commuting of the type \((P)\).
\(c\). For \(R=2\), pair \((f, g)\) is R-weakly commuting of type \((A_f)\), and R-weakly commuting of type \((A_g)\) and not R-weakly commuting of type \((P)\)(for this take \(x = \frac{3}{4}\)).
Example 1.5. Consider $X=\left[\frac{1}{2},2\right]$. Let us define self maps $f$ and $g$ by

$$fx=\frac{x+1}{3}, gx=\frac{x+2}{5}.$$ We calculate the following:

$$d(fx, gx)=\frac{2x-1}{15}, d(fgx, gfx)=0, d(fgx, ggx)=\frac{2x-1}{75}, d(gfx, ffx)=\frac{2x-1}{45} \text{ and }$$

$$d(ffx, ggx)=\frac{8}{225}(2x-1).$$

Now we conclude the following:

The pair $(f,g)$ is $R$-weakly commuting for all positive real numbers.

For $R \geq \frac{8}{15}$, it is $R$-weakly commuting of type $(A_f)$, $R$-weakly commuting of type $(A_g)$ and $R$-weakly commuting of type $(P)$.

For $\frac{1}{3} \leq R < \frac{8}{15}$, it is $R$-weakly commuting of type $(A_g)$ and $R$-weakly commuting of type $(P)$.

For $\frac{1}{5} \leq R < \frac{1}{3}$, it is $R$-weakly commuting of type $(A_f)$ but not $R$-weakly commuting of type $(P)$.

Moreover, such mappings commute at their coincidence points. It is also obvious that $f$ and $g$ can fail to be point wise $R$-weakly commuting only if there exists some $x$ in $X$ such that $fx=gx$ but $fgx \neq gfx$, that is, only if they posses a coincidence point at which they do not commute. Therefore, the notion of point wise $R$-weak commutativity type mapping is equivalent to commutativity at coincidence points.

2. Main Results

In 1984, Wang, Li, Gao and Iseki [14] and Rhoades [12] proved some fixed point theorems for expansion mappings, which correspond to some contractive mappings in metric spaces.

In a recent paper, Daffer and Kaneko [3] proved the following fixed point theorem.

Theorem 2.1. Let $(X,d)$ be a complete metric space. Let $f$ be a surjective self map and $g$ be injective self map of $X$ which satisfy the following conditions:

There exists a number $1 < q$ such that
(2.1) \( d(f_x, f_y) \geq q d(g_x, g_y) \) for each \( x, y \) in \( X \), then \( f \) and \( g \) have a unique common fixed point.

The following result is an extension of the Theorem 2.1 to compatible maps by Rhoades [12].

**Theorem 2.2.** Let \((X, d)\) be a complete metric space. Let \( f \) and \( g \) be compatible self maps of \( X \) satisfying conditions (2.1) and \( g(X) \subseteq f(X), f \) continuous. Then \( f \) and \( g \) have a unique common fixed point.

In 2008 Kumar [8] generalized Theorems 2.2 to weakly compatible maps in the form of following:

**Theorem 2.3.** Let \((X, d)\) be a complete metric space. Let \( f \) and \( g \) be weakly compatible self maps of \( X \) satisfying conditions (2.1) and the following \( g(X) \subseteq f(X) \).

If one of the subspaces \( g(X) \) or \( f(X) \) is complete, then \( f \) and \( g \) have a unique common fixed point.

For the sake of the convenience of readers we are giving here proof.

**Proof.** Let \( x_0 \in X \). Since \( g(X) \subseteq f(X) \), choose \( x_1 \in X \) such that \( f x_1 = g x_0 \). In general, choose \( x_{n+1} \) such that \( y_n = f x_{n+1} = g x_n \). Then, from (2.1),

\[
d(g x_n, g x_{n+1}) \leq d(f x_n, f x_{n+1})/q,
\]

which implies that \( \{g x_n\} \) is Cauchy, see [12], hence convergent. Call the limit \( z \), then

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z
\]

Since \( f(X) \) is complete, so there exists a point \( p \in X \) such that \( f p = z \).

Now from (2.1),

\[
d(g p, g x_n) \leq 1/q d(f p, f x_n)
\]

Proceeding limit as \( n \to \infty \), we have

\[
d(g p, z) \leq 1/q d(f p, z),\text{which implies that } g p = z.\text{ Therefore, } g p = f p = z.\text{ Since } f \text{ and } g \text{ are weakly compatible, therefore,}

\[
f g p = g f p, \text{i.e., } f z = g z.
\]

Now we show that \( z \) is a fixed point of \( f \) and \( g \). From (2.1)

\[
d(f z, f x_n) \geq q d(g z, g x_n)\text{. Proceeding limit as } n \to \infty, \text{ we have}
\]

\[
d(f z, z) \geq q d(g z, z),\text{which implies that } f z = z.\text{ Hence } z \text{ is a fixed point of } f \text{ and } g .\text{ Uniqueness follows easily from (2.1).}

Now we give an example in support of theorem.

**Example 2.1.** Let \( X = [0,1] \) with the usual metric i.e., \( d(x, y) = |x - y| \). Define mappings \( f, g : X \to X \) \( f(x) = x/2, g(x) = x/6 \). Then \( g(X) \subseteq f(X) \).

Moreover, \( d(f x, f y) = 1/2 \{ |x - y| \} \geq q/6 \{ d(g x, g y) \} \) for \( 1 < q < 3 \) and (2.1) is satisfied. However, maps are weakly compatible at \( x = 0 \) and zero is the unique
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common fixed point of f and g. Thus all the conditions of the Theorem 2.3 are satisfied.

Now we prove the following results after relaxing the condition of completeness of the space:

Theorem 2.4. Let (X,d) be a metric space. Let f and g be weakly compatible self maps of X satisfying conditions (2.1) and the following
\[ g(X) \subseteq f(X). \]
If one of the subspaces g(X) or f(X) is complete, then f and g have a unique common fixed point.

Proof: From the proof of Theorem 2.3, we conclude that \( \{ y_n \} \) is a Cauchy sequence in X. Now suppose that f(X) is a complete subspace of X, then the subsequence of \( \{ y_n \} \) must get a limit in f(X). Call it be v and u \( \in f^{-1}v \). Then fv=uv. As \( \{ y_n \} \) is a Cauchy sequence containing a convergent subsequence, therefore the sequence \( \{ y_n \} \) also converges implying thereby the convergence of subsequence of the convergent sequence. On setting x = v and y = x_n in (2.1),
\[ d(gv, gx_n) \leq d(fv, fx_n)/q, \]
which implies that
\[ fv = gv = u, \]
which shows that the pair (f,g) has a point of coincidence.
Since f and g are weakly compatible, therefore,
\[ fgv = gfv, \text{ i.e., } fu = gu. \]
Now we show that u is a fixed point of f and g. From (2.1)
\[ d(fu, fx_n) \geq q d(gu, gx_n). \]
Proceeding limit as \( n \to \infty \), we have
\[ d(fu, u) \geq q d(fu, u), \]
which implies that fu = u. Hence u is a fixed point of f and g. Uniqueness follows easily.

Theorem 2.5. Theorem 2.4 remains true if a ‘weakly compatible property is replaced by any one (retaining the rest of the hypotheses) of the following:
(i) R-weakly commuting property,
(ii) R-weakly commuting property of type (A_4),
(iii) R-weakly commuting property of type (A_g),
(iv) R-weakly commuting property of type (P),
(v) weakly commuting property.

Proof. Since all the conditions of Theorem 2.4 are satisfied, then the existence of coincidence points for both the pairs is insured. Let x be an arbitrary point of coincidence for the pair (f, g), then using R-weak commutativity one gets
\[ d(fgx, gfx, ) \leq R d (fx, gx, ) = 0, \]
which amounts to say that fgx = gfx. Thus the pair (f, g) is weakly compatible. Now applying Theorem 2.4, one concludes that f and g have a unique common fixed point.
In case \((f, g)\) is an R-weakly commuting pair of type \((A_f)\), then
\[
d(fgx, g^2x) \leq d(fx, gx) = 0,
\]
which amounts to say that \(fgx = g^2x\). Now
\[
d(fgx, gfx) \leq d(fgx, g^2x) + d(g^2x, gfx) = 0 + 0 = 0,
\]
yielding thereby \(fgx = gfx\).

In case \((f, g)\) is an R-weakly commuting pair of type \((A_g)\), then
\[
d(gfx, f^2x) \leq d(fx, gx) = 0,
\]
which amounts to say that \(gfx = f^2x\). Now
\[
d(gfx, gfx) \leq d(gfx, f^2x) + d(f^2x, gfx) = 0 + 0 = 0,
\]
yielding thereby \(fgx = gfx\).

Similarly, if pair is R-weakly commuting mappings of type \((P)\) or weakly commuting, then \((f, g)\) also commutes at their points of coincidence. Now in view of Theorem 2.4, in all four cases \(f\) and \(g\) have a unique common fixed point. This completes the proof.

As an application of Theorem 2.4, we prove a common fixed point theorem for two finite families of mappings which runs as follows:

**Theorem 2.6.** Let \(\{f_1, f_2, \ldots, f_m\}\) and \(\{g_1, g_2, \ldots, g_n\}\) be two finite families of self-mappings of a metric space \((X, d)\) such that \(f = f_1f_2 \ldots f_m, g = g_1g_2 \ldots g_n\) satisfy the condition (2.1) with \(g(X) \subset f(X)\).

If one of \(f(X)\) or \(g(X)\) is a complete subspace of \(X\), then \(f\) and \(g\) have a point of coincidence,

Moreover, if \(f_if_j = f_jf_i, g_ig_k = g_kg_i\) for all \(i, j \in I_1 = \{1, 2, \ldots, m\}, k, l \in I_2 = \{1, 2, \ldots, n\}\), then (for all \(i \in I_1, k \in I_2\)) \(f_i\) and \(g_k\) have a common fixed point.

**Proof.** The conclusions is immediate i.e., \(f\) and \(g\) have a point of coincidence as \(f\) and \(g\) satisfy all the conditions of Theorem 2.4. Now appealing to component wise commutativity of various pairs, one can immediately prove that \(fg = gf\) hence, obviously pairs \((f, g)\) is coincidentally commuting. Note that all the conditions of Theorem 2.4 are satisfied ensuring the existence of a unique common fixed point, say \(z\). Now one need to show that \(z\) remains the fixed point of all the component maps. For this consider
\[
\bar{f}(f_iz) = ((f_1f_2 \ldots f_m)f_iz) = (f_1f_2 \ldots f_{m-1})(f_mf_iz) = (f_1 \ldots f_{m-2})(f_{m-1}f_mz) = \ldots = f_1f_2f_3f_4 \ldots f_mz = f_1f_2f_3f_4 \ldots f_mz
\]
Similarly, one can show that
\[
\bar{f}(g_iz) = g_1(f_iz) = g_iz, \ g(g_iz) = g_1(g_iz) = g_iz\quad and\quad g(f_iz) = f_1(g_iz) = f_iz,
\]
which show that (for all $i$ and $k$) $f_iz$ and $g_kz$ are other fixed points of the pair $(f, g)$.

Now appealing to the uniqueness of common fixed points of both pairs separately, we get

$$z = f_iz = g_kz,$$

which shows that $z$ is a common fixed point of $f_i$, $g_k$ for all $i$ and $k$.

By setting $f = f_1 = f_2 = \ldots = f_m$, $g = g_1 = g_2 = \ldots = g_n$, in Theorem 2.4 we deduces the following:

**Corollary 3.1.** Let $f$ and $g$ be two self-mappings of a metric space $(X, d)$ such that $f_m$ and $g_n$ satisfy the condition (2.1). If one of $f_m(X)$ or $g_n(X)$ is a complete subspace of $X$, then $f$, and $g$ have a unique common fixed point provided $(f, g)$ commute.

**References**

[1] R. Chugh and S. Kumar, Common fixed points for weakly compatible maps


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