Weak Primary Submodules of Multiplication Modules and Intersection Theorem

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Abstract

As a new generalization of the notion of primary ideals to multiplication modules, we introduce a weak primary submodule in a multiplication module. It is shown that if $Q$ is a proper submodule of multiplication $R$-module $M$ such that $M/Q$ or $\sqrt{Q}$ is finitely generated, then $Q$ is primary if and only if $Q$ is weak primary. Finally, we state and prove an extension of Krull’s Intersection Theorem to multiplication modules.

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1 Introduction

Throughout this paper all rings will be commutative with nonzero identity and all modules will be unitary. Let $R$ be a ring and $M$ be an $R$-module. For every submodule $N$ of $M$, we will be denoted the annihilator of factor module $M/N$ by $(N :_R M)$ (in fact; $(N :_R M) = \{ r \in R \mid rM \subseteq N \}$). Also $(0 :_R M)$ is denoted by $Ann_R(M)$. An element $r \in R$ is called a zero-divisor on $M$ if $rm = 0$, for some nonzero $m \in M$. The set of all zero-divisors of $R$ on $M$ is denoted by $Zdv_R(M)$. In this time, we recall the concept multiplication module
and state some properties of them. For basic properties of a multiplication module one may refer to [2], [3] and [7].

**Definition 1.1.** Let $R$ be a ring and $M$ be an $R$-module. Then $M$ is called a multiplication module if for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$. In this case we can take $I = (N :_R M)$.

Let $M$ be a multiplication $R$-module and let $N$ and $K$ be submodules of $M$ with $N = I_1M$ and $K = I_2M$ for some ideals $I_1$ and $I_2$ of $R$. The product of $N$ and $K$ denoted by $NK$ is defined by $NK = I_1I_2M$. The product of $N$ and $K$ is independent of ideals $I_1$ and $I_2$, see [1; Theorem 3.4]. Moreover, it is clear that $NK$ is a submodule of $M$ and contained in $N \cap K$. Also, if $N \subseteq K$ then $NL \subseteq KL$, for every submodule $L$ of $M$. Furthermore, for every elements $m, m' \in M$, we show by $mm'$ the product of two submodules $Rm$ and $Rm'$. Therefore product $mm'$ is a submodule of $M$.

We recall that a submodule $N$ of $M$ is called nilpotent if $N^k = 0$, for some positive integer $k$, where $N^k$ means the product of $N$, $k$ times. Also an element $m \in M$ is called nilpotent if submodule $Rm$ is nilpotent (see [1, Theorem 3.4]). An element $m \in M$ is called a zero-divisor over $M$ if $mm' = 0$, for some nonzero $m' \in M$. The set of all zero-divisors of $M$ over $M$ is denoted by $Zdv_M(M)$. By [4, Proposition 2.8], $R.Zdv_M(M) = Zdv_R(M)M$.

Let $M$ be an $R$-module. A proper submodule $Q$ of $M$ is said to be prime (resp. primary) submodule if $am \in Q$, where $a \in R$ and $m \in M \setminus Q$, then $aM \subseteq Q$ (resp. $a^k M \subseteq Q$ for some positive integer $k$). We say that $M$ is a prime (resp. primary) module if zero submodule of $M$ is prime (resp. primary) submodule of $M$. The radical of a proper submodule $N$ of $M$, denoted by $\sqrt{N}$, is defined in [3] to be the intersection of all prime submodules of $M$ containing $N$.

**Theorem 1.2.** [1; Theorem 3.13] Let $N$ be a submodule of a multiplication $R$-module $M$. Then $\sqrt{N} = \{m \in M \mid m^k \subseteq N \text{ for some } k \geq 0\}$.

We will denote by $\text{rad}(I)$ the radical of an ideal $I$ of a ring $R$. If $Q$ is a primary ideal of a ring $R$, it is well-known that $\sqrt{Q}$ is a prime ideal of $R$. However, in the module case, $Q$ a primary submodule dose not necessarily imply that $\sqrt{Q}$ is a prime submodule (see [8; Theorem 1.9 and Example 1.11] for more details). Finally we mention the other results that will be used in the sequel.

**Theorem 1.3.** [3; Corollary 2.11] Let $N$ be a proper submodule of a multiplication $R$-module $M$. Then $\sqrt{N} = IM$, where $I = \text{rad}((N :_R M))$.

In this paper, for simplicity, we will denote $\text{rad}(\text{Ann}_R(M))$ by $\text{nil}_R(M)$, for every $R$-module $M$. Therefore $\sqrt{0} = (\text{nil}_R(M))M$, for each multiplication
2 Weak primary Submodules

In this section, we introduce special submodules of a multiplication module over commutative rings and we state equivalent statements with them.

**Definition 2.1.** Let $R$ be a commutative ring and $M$ be a multiplication $R$-module. A proper submodule $Q$ of $M$ is called a weak primary submodule if $mn \subseteq Q$, where $m, n \in M$ and $m \notin Q$, then $n^k \subseteq Q$ for some natural element $k$. Also $M$ is called a weak primary module if the zero submodule of $M$ is a weak primary submodule of $M$.

It is easy to check that a nonzero $R$-module $M$ is primary if and only if $Zdv_R(M) = \text{nil}_R(M)$. The next Proposition is similar to Theorem 3.6 of [4] for a weak primary multiplication $R$-module.

**Proposition 2.2.** Let $M$ be a nonzero multiplication $R$-module. Then the following statements are equivalent:

1) $M$ is a weak primary module;

2) $Zdv_M(M) = \sqrt{0}$;

3) if $mn = 0$, where $m, n \in M$, then $m = 0$ or $n \in \sqrt{0}$;

4) for every submodules $N_1$ and $N_2$ of $M$, if $N_1N_2 = 0$, then $N_1 = 0$ or $N_2 \subseteq \sqrt{0}$.

**Proof.** (1) $\Rightarrow$ (2). Let $M$ be a weak primary module. First suppose $n \in Zdv_M(M)$, then $mn = 0$, for some nonzero $m \in M$ and so $n^k = 0$, for some positive integer $k$. Therefore $n \in \sqrt{0}$, by Theorem 1.2.

Now let $n \in \sqrt{0}$. Then by Theorem 1.2, $n^k = 0$, for some positive integer $k$. We assume that $k$ is the least positive integer such that $n^k = 0$. If $k = 1$, then $n = 0 \in Zdv_M(M)$. If $k > 1$, then there is a nonzero element $m \in n^{k-1}$. Since $mn \subseteq n^k = 0$, so $n \in Zdv_M(M)$. Thus $Zdv_R(M) = \sqrt{0}$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1). Let $mn = 0$, for some $m, n \in M$. Then $m = 0$ or $n \in \sqrt{0}$ and so $n^k = 0$, for some positive integer $k$, by Theorem 1.2. Therefore $\{0\}$ is a weak primary submodule of $M$ and so $M$ is a weak primary module.

Let $M$ be an $R$-module, $N$ be a submodule of $M$ and $\pi : M \rightarrow \frac{M}{N}$ is natural homomorphism. For every submodule $K$ (resp. element $m$) of $M$, the
submodule \( \pi(K) \) (resp. the element \( \pi(m) \)) is denoted by \( \overline{K} \) (resp. \( \overline{m} \)).

**Proposition 2.3.** Let \( M \) be a multiplication \( R \)-module, \( N \) be a proper submodule of \( M \) and \( \pi : M \to \frac{M}{N} \) is natural homomorphism. We use the notation above, then the following statements are holds:

1) for every submodule \( N_1 \) and \( N_2 \) of \( M \), \( \overline{N_1N_2} = 0 \) if and only if \( N_1N_2 \subseteq N \).

2) for every \( m, n \in M \), \( \overline{mn} = 0 \) if and only if \( mn \subseteq N \).

**Proof.** It is sufficient to prove (1).

1. \( (\Rightarrow) \) It is evident that \( \frac{M}{N} \) is a multiplication \( R \)-module. If \( N_1 = I_1M \), \( N_2 = I_2M \) and \( N = JM \), for some ideals \( I_1, I_2 \) and \( J \) of \( R \), then

\[
\overline{N_1} = I_1 \frac{M}{N} = I_1 \frac{M}{N} = \left( I_1 + J \right) \frac{M}{N}.
\]

Thus if \( \overline{N_1N_2} = 0 \), then \( \left( \frac{I_1I_2 + J}{N} \right) \frac{M}{N} = \overline{N_1N_2} = 0 \), and so \( N_1N_2 + N = (I_1I_2 + J)M = N \). Therefore \( N_1N_2 \subseteq N \).

\( (\Leftarrow) \) This is proved similarly. \( \square \)

The next result follows immediately, by Proposition 2.2 and Proposition 2.3.

**Corollary 2.4.** Let \( N \) be a proper submodule of a multiplication \( R \)-module \( M \). Then \( N \) is a weak primary submodule of \( M \) if and only if \( R \)-module \( \frac{M}{N} \) is a weak primary module.

Furthermore we can infer any results proved about zero submodule of \( M \); to similar results for proper submodules of \( M \).

**Corollary 2.5.** Let \( M \) be a multiplication \( R \)-module and \( N \) be a proper submodule of \( M \). Then the following statements are equivalent:

1) \( N \) is a weak primary submodule;

2) \( Zdv_{\frac{M}{N}}(M/N) = \{ \overline{m} \in \frac{M}{N} \ | \ m \in \sqrt{N} \} \);

3) for every \( m, n \in M \), if \( mn \subseteq N \), then \( m \in N \) or \( n \in \sqrt{N} \).

4) for every submodules \( N_1 \) and \( N_2 \) of \( M \), if \( N_1N_2 \subseteq N \), then \( N_1 \subseteq N \) or \( N_2 \subseteq \sqrt{N} \)

A multiplicative \( R \)-module \( M \) is called \( w \)-primary compatible if its weak primary submodules and its primary submodules coincide. The next proposition explore that finitely generated multiplication modules are \( w \)-primary
compatible.

**Theorem 2.6.** Let $M$ be a multiplication $R$-module. If $M$ is a primary module, then $M$ is a weak primary module. If $M$ is finitely generated, then the converse is true.

**Proof.** Suppose that $M$ be a primary module and $mn = 0$, for some $m, n \in M$ and $m \neq 0$. There are ideals $I$ and $J$ of $R$ such that $Rm = IM$ and $Rn = JM$. Then $IJ \subseteq \text{Ann}_R(M)$ and $I \not\subseteq \text{Ann}_R(M)$. Since $M$ is primary, then $\text{Ann}_R(M)$ is a primary ideal and so $J \subseteq \text{nil}_R(M)$. By Theorem 1.3, $n \in JM \subseteq \sqrt{0}$ and so by Proposition 2.2, $M$ is a weak primary module.

Now, let $M$ be a weak primary finitely generated module and $rm = 0$, for some $r \in R$ and $0 \neq m \in M$. There is an ideal $I$ of $R$ such that $Rm = IM$. Then $(IM)(IM) = (IIM) = IIM = rIM = rRM = 0$. By Proposition 2.2 and Theorem 1.3, $rM \subseteq \sqrt{0} \subseteq (\text{nil}_R(M))M$. Since $M$ is finitely generated, $r \in \text{nil}_R(M)$, by [5; Theorem 6.6], and so $r^k M = 0$, for some positive integer $k$. Therefore $M$ is a primary module. □

We deduce the following from Theorem 2.6 and Corollary 2.4.

**Corollary 2.7.** Let $N$ be a proper submodule of a multiplication $R$-module $M$. If $M/N$ is finitely generated, then $N$ is a weak primary submodule if and only if $N$ is a primary submodule.

It is well-known that if $I$ is an ideal of a commutative ring $R$ such that $\text{rad}(I)$ is finitely generated, then $\text{rad}(I)^k \subseteq I$, for some positive integer $k$ (see [6; Lemma 8.21]). The next Proposition extend this result to submodules of multiplication modules.

**Proposition 2.8.** Let $M$ be a multiplication $R$-module and $N$ be a submodule of $M$. If $\sqrt{N}$ is finitely generated, then $(\sqrt{N})^k \subseteq N$, for some positive integer $k$.

**Proof.** There are elements $m_1, m_2, ..., m_l \in M$ and ideals $I_1, I_2, ..., I_l$ of $R$ such that $\sqrt{N} = Rm_1 + \cdots + Rm_l$ and $Rm_j = I_j M$, for each $j$, $(1 \leq j \leq l)$. By Theorem 1.2, for every $j$, there is a positive integer $k_j$ such that $m_j^{k_j} \subseteq N$. Let $k = \sum_{j=1}^{l} (k_j - 1) + 1$. Then

$$\sqrt{N}^k = (\sqrt{N})^k = (Rm_1 + Rm_2 + ... + Rm_l)^k = (I_1 M + I_2 M + ... + I_l M)^k = \sum_{s_1 + ... + s_l = k} (I_1 M)^{s_1} (I_2 M)^{s_2} ... (I_l M)^{s_l} = \sum_{s_1 + ... + s_l = k} m_1^{s_1} m_2^{s_2} ... m_l^{s_l} \subseteq N. \quad \Box$$
Lemma 2.9. Let $M$ be a multiplication $R$-module and $N$ be a submodule of $M$. For every positive integer $k$, $(\sqrt{N})^k \subseteq N$ if and only if $(\text{rad}(N :_R M))^k \subseteq (N :_R M)$.

Proof. By Theorem 1.3, $(\sqrt{N})^k \subseteq N$ if and only if $(\text{rad}(N :_R M))^k M \subseteq N$ and this equivalent with $(\text{rad}(N :_R M))^k \subseteq (N :_R M)$. □

Corollary 2.10. Let $M$ be a multiplication $R$-module. If the ideal $\text{nil}_R(M)$ or the submodule $\sqrt{0}$ is finitely generated, then $\sqrt{0}$ is nilpotent submodule.

The next Theorem gives the other condition that weak primary submodules and primary submodules coincide.

Theorem 2.11. Let $M$ be a multiplication $R$-module and $N$ be a submodule of $M$. If $\sqrt{0}$ is nilpotent submodule, then the following statements are equivalent:

1) $M$ is a primary module;

2) $M$ is a weak primary module;

3) for every submodules $N_1$ and $N_2$ of $M$, if $N_1N_2 = 0$, then $N_1 = 0$ or $N_2^k = 0$, for some positive integer $k$.

Proof. (1) $\Rightarrow$ (2) It follows from Proposition 2.6.

(2) $\Rightarrow$ (3). Let $N_1N_2 = 0$ and $N_1 \neq 0$, for submodules $N_1$ and $N_2$ of $M$. By Proposition 2.2, $N_2 \subseteq \sqrt{0}$ and so $N_2^k = 0$, for some positive integer $k$, because $\sqrt{0}$ is nilpotent.

(3) $\Rightarrow$ (1). Let $rm = 0$, for some $r \in R$ and $0 \neq m \in M$. There is an ideal $I$ of $R$ such that $Rm = IM$. Thus $(rM)(IM) = (\langle r \rangle M)(IM) = \langle r \rangle IM = \langle r \rangle Rm = rRm = 0$. Since $IM \neq 0$, then $r^k M = (rM)^k = 0$, for some positive integer $k$. Therefore $M$ is primary. □

The next result follows of Corollary 2.4 and Theorem 2.11.

Corollary 2.12. Let $N$ be a proper submodule of a multiplication $R$-module $M$. If $\sqrt{N}$ is finitely generated, then $N$ is a weak primary submodule if and only if $N$ is a primary submodule.
3 Intersection Theorem in multiplication modules

Recall that in a multiplication module every primary submodule is weak primary. Then we can use properties of weak primary submodules for primary submodules. For example the next theorem extend a well-known theorem in commutative rings to multiplication modules.

**Theorem 3.1.** Let $M$ be a multiplication Noetherian $R$-module and $N$ be a submodule of $M$. If $L = \bigcap_{k=1}^{\infty} N^k$, then $LN = L$.

**Proof.** If $N = M$, then the claim is clear, and so we assume that $N \neq M$. Since $LN \subseteq L \subseteq N$, then $LN$ is also a proper submodule. But $M$ is Noetherian, hence by [6; Exercise 9.31], $N$ has a primary decomposition. Let $LN = Q_1 \cap Q_2 \cap ... \cap Q_l$ be a minimal primary decomposition of $LN$. It is sufficient to prove that $L \subseteq Q_i$, for all $i$ (1 $\leq$ $i$ $\leq$ $l$). Suppose that, in contrary, we have $L \nsubseteq Q_j$, for some $j$, with 1 $\leq$ $j$ $\leq$ $l$, so that there exists an element $m \in L$ such that $m \notin Q_j$. Since $mN \subseteq LN \subseteq Q_j$ and $Q_j$ is weak primary submodule, by Theorem 2.6, then $N \subseteq \sqrt{Q_j}$, by Corollary 2.5.

On the other hand, Since $M$ is Noetherian, then $\sqrt{Q_j}$ is finitely generated and so $(\sqrt{Q_j})^k \subseteq Q_j$, for some positive integer $k$, by Proposition 2.8. Hence $L \subseteq N^k \subseteq (\sqrt{Q_j})^k \subseteq Q_j$. This is a contradiction. Therefore $L \subseteq Q_i$, for all $i$, and so $LN = L$. $\square$

Let $M$ be a Noetherian $R$-module. We will denote the intersection of all maximal submodule of $M$ by $\text{Rad}_R(M)$. By the above Theorem, we can easily be extended Krull’s Intersection Theorem to multiplication modules.

**Theorem 3.2** Let $M$ be a multiplication Noetherian $R$-module and $N$ be a submodule of $M$. If $N \subseteq \text{Rad}_R(M)$, then $\bigcap_{k=1}^{\infty} N^k = 0$.

**Proof.** Set $L = \cap_{k=1}^{\infty} N^k$. By Theorem 3.1, $LN = L$. Moreover there are ideals $I$ and $J$ of $R$ such that $N = IM$ and $L = JM$. Since $L = LN$, then $L = (JM)(IM) = IJM = IL$. It is clear that $M$ is a faithful Noetherian $\overline{R}$-module, where $\overline{R} = \frac{R}{\text{Ann}(M)}$. If $f : R \rightarrow \overline{R}$ is canonical homomorphism and $\overline{I}$ is the image of $I$, then $\overline{IL} = L$. Since $M$ is faithful, then $\text{Rad}_\overline{R}(M) = \text{Jac}(\overline{R})M$, by [3; Theorem 2.7]. But $IM = N \subseteq \text{Rad}_R(M)$, then $\overline{IM} \subseteq \text{Jac}(\overline{R})M$ and so $\overline{I} \subseteq \text{Jac}(\overline{R})$, by [3; Theorem 3.1]. Finally, from $\overline{IL} = L$, we conclude that $L = 0$, by Nakayama Lemma. $\square$

**Corollary 3.3.** Let $M$ be a multiplication Noetherian $R$-module and $N$ be
the only maximal submodule of $M$. then $\cap_{k=1}^{\infty} N_k = 0$.

References


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