Dual Integral Equations and Singular Integral Equations for Helmholtz Equation

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Abstract

Paper is devoted for Solving a Helmholtz equation in cylindrical coordinates under mixed discontinuous boundary conditions of the third kind (Robin's boundary conditions) for a semi-space and for infinite solid plate. A dual integral equations (DIE) method were reduce the given mixed boundary value problem to a Fredholm singular integral equation of the second kind.

Keywords: Dual integral equations, mixed boundary conditions, Helmholtz equation

1. Introduction

Helmholtz partial differential equation arises in the study of mathematical physics equations with several coordinate systems. In this paper we present a Helmholtz equation in axial symmetrical cylindrical coordinates under mixed discontinuous boundary conditions of the third kind acted on the surface of semi-infinite space and on the surface of infinite plate, the solution of the of the Helmholtz mixed boundary value problem is obtained with the aid of the DIE method, the DIE are reduced to some type of Fredholm singular integral equation of the second kind. Theory related to mixed problem which discussed in this paper is due to Uflyand [8], the technique was used with details to investigate the solution of some DIE for a Laplace's equation with some mixed boundary value problems.
2. Formulation of the problem

Suppose that for instance we wish to determine the function \( u = u(r, z) \), satisfies Helmholtz partial differential equation in a axial symmetrical semi-space of a solid cylinder subject to the mixed boundary conditions of the third kind acted on the level surface \( z = 0 \) of the semi-space in cylindrical coordinates

\[
a_i u_z + b_i u = f_i(r), \quad r \in (0, l)
\]

\[
a_2 u_z + b_2 u = f_2(r), \quad r \in (l, \infty)
\]

\( a_i, b_i, i = 1, 2 \) are constants \( u_z = \partial u / \partial z \), \( f_i(r), i = 1, 2 \) known continuous functions accepted Hankel integral transform, \( l \) is the line of discontinuity (the radius of the disk). Separation variables to the Helmholtz equation \( \nabla^2 u + k^2 u = 0 \), \( k \) is a constant, we assume that \( u(r, z) \) is bounded at zero and infinity, a general solution becomes

\[
u(r, z) = \int_0^\infty C(p) J_0(pr) \exp(-z \sqrt{p^2 + k^2}) dp
\]

Use a mixed conditions (2.1),(2.2) for (2.3), we obtain a pair of DIE for determination the unknown function \( C(p) \)

\[
\int_0^\infty C(p) J_0(pr) (a_1 \sqrt{p^2 + k^2} + b_1) dp = f_1(r), \quad r \in (0, l)
\]

\[
\int_0^\infty C(p) J_0(pr) (a_2 \sqrt{p^2 + k^2} + b_2) dp = f_2(r), \quad r \in (l, \infty)
\]

The DIE (2.4) and (2.5) should be reduced to the standard form

\[
\int A(p) J_0(pr) [1 - g(p)] dp = F(r), \quad r \in (0, l)
\]

\[
\int A(p) J_0(pr) dp = 0, \quad r \in (l, \infty)
\]

Where \( g(p) = \frac{(a_2 - a_1) \sqrt{p^2 + k^2} + (b_2 - b_1)}{a_1 \sqrt{p^2 + k^2} + b_1} \)

\[
F(r) = f_1(r) - \int_0^\infty p v(p) J_0(pr) [1 - g(p)] dp
\]

\[
A(p) = C(p) J_0(pr) (a_2 \sqrt{p^2 + k^2} + b_2)
\]

\[
\int v(p) J_0(pr) dp = f_2(r) \quad \text{and} \quad v(p) = \int_0^\infty r f_2(r) J_0(pr) dr
\]

Rewrite (2.6) as

\[
\int A(p) J_0(pr) dp = F(r) + \int A(p) J_0(pr) g(p) dp
\]
Replacing a function \( A(p) \) by another unknown continuous function \( \phi(t) \) with the help of the relation [8]

\[
A(p) = p \int_0^l \phi(t) \cos(pt) dt
\]  

(2.9)

For instance put \( g(p) = 0 \) in (2.8), then use the Hankel's inverse transform [9]

\[
A(p) = p \int_0^l rF(r)J_0(pr) dr
\]  

(2.10)

Use the known relation [8,9]

\[
J_0(pr) = \frac{2}{\pi} \int_0^l \cos(pt) \sqrt{r^2 - t^2} dt
\]  

(2.11)

Put (2.11) into (2.10) then interchange the order of integration we have

\[
A(p) = \frac{2}{\pi} \int_0^l \cos(pt) \left( \int_0^l \frac{rF(r)dr}{\sqrt{r^2 - t^2}} \right) dt
\]  

(2.12)

Now compare (2.12) and (2.9), we discover that

\[
\phi(t) = \frac{2}{\pi} \int_0^l \frac{rF(r)dr}{\sqrt{r^2 - t^2}} dt
\]  

(2.13)

Now when \( g(p) \neq 0 \) in (2.8), expressions (2.13) and (2.9) yield a Fredholm singular integral equation of the second kind to determine the unknown function \( \phi(t) \)

\[
\phi(x) = \frac{2}{\pi} H(x) + \int_0^l K(x,t) \phi(t) dt
\]  

(2.14)

Where the free term \( H(x) = \frac{1}{\pi} \int_0^l \frac{rF(r)}{\sqrt{r^2 - x^2}} dr \) and

and the kernel \( K(x,t) = \frac{1}{\pi} \int_0^l \frac{rdr}{\sqrt{r^2 - x^2}} \int_0^l pg(p)J_0(pr) \cos(pt) dp \).

The kernel and the free term should be satisfied the properties [4, 6]

\[
\int_0^l |H(x)| dx < \infty , \int_0^l |K^2(x,t)| dx dt < \infty
\]  

(2.15)

In similar manner, the DIE involving mixed boundary conditions to the Helmholtz equation for unbounded plate high \( h \) when on the surface \( z = 0 \) a boundary conditions (2.1) , (2.2) are given, whereas on another surface \( z = h \) a homogeneous unmixed boundary condition is

\[
u(r, h) = 0
\]  

(2.16)

Separation variables to the Helmholtz equation in cylindrical coordinates under the assumption that \( u \) is bounded at zero and infinity, the general solution is

\[
u(r, z) = \int_0^\infty C(p)J_0(pr) \frac{sh[(h-z)\sqrt{p^2 + k^2}]}{ch[h\sqrt{p^2 + k^2}]} dp
\]  

(2.17)
sh(x), ch(x) sine and cosine hyperbolic functions, Apply the mixed (2.1), (2.2) to (2.17), we get a DIE for determination $A(p)$

$$\int_0^\infty A(p) J_0(pr)[1 - g(p)]dp = F(r), \quad r \in (0, l) \quad (2.18)$$

$$\int_0^\infty A(p) J_0(pr) \, dp =, \quad r \in (l, \infty) \quad (2.19)$$

$$g(p) = \frac{(a_2 - a_1) \sqrt{p^2 + k^2} + (b_2 - b_1) \text{th}(h \sqrt{p^2 + k^2})}{a_2 \sqrt{p^2 + k^2} + b_2 \text{th}(h \sqrt{p^2 + k^2})},$$

$$A(p) = C(p) \left( a_2 \sqrt{p^2 + k^2} + b_2 \text{th}(h \sqrt{p^2 + k^2}) \right)$$

$\text{th}(x)$ is the tangent hyperbolic function. The last DIE (2.18) and (2.19) should be reduced to a Fredholm singular integral equation of the form (2.14) by applying the same technique mentioned above. Integral equation (2.14) can be solved numerically with the use of some math software [3,7]. As $h \to \infty$, the DIE (2.18) and (2.19) correspond the DIE involve a semi-space (2.6) and (2.7). Method of treating DIE illustrated in this paper can be used widely to solve several types of mixed boundary value problems for heat and wave equations in different coordinates. In particular for some of a constants $a_i, b_i, i=1,2$ is zero, a dual equations (2.6) and (2.7) were discussed with the use of some discontinuous integrals, integral transform and operational calculus methods[1, 2, 5]

References


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