On the $L$-Order and $L$-Type of Wronskians

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Abstract

In the paper we study the relationship between the $L$-order ($L$-type) of a transcendental meromorphic function and that of a wronskian generated by it.

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1 Introduction, Definitions and Notations.

We denote by $C$ the set of all finite complex numbers. Let $f$ be a meromorphic function defined on $C$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [4] and [1]. In the sequel we use the following notation:

$log^k x = log(log^{k-1} x)$ for $k = 1, 2, 3, \ldots$ and $log^0 x = x$.

The following definitions are well known:

**Definition 1.** The order $\rho_f$ and lower order $\lambda_f$ of a meromorphic function $f$ is defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$
If \( f \) is entire then
\[
\rho_f = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log r}.
\]

**Definition 2.** The hyper order \( \rho_f \) and hyper lower order \( \lambda_f \) of a meromorphic function \( f \) is defined as
\[
\tilde{\rho}_f = \limsup_{r \to \infty} \frac{\log^2 T(r, f)}{\log r} \quad \text{and} \quad \tilde{\lambda}_f = \liminf_{r \to \infty} \frac{\log^2 T(r, f)}{\log r}.
\]

If \( f \) is entire then one can easily verify that
\[
\tilde{\rho}_f = \limsup_{r \to \infty} \frac{\log^3 M(r, f)}{\log r} \quad \text{and} \quad \tilde{\lambda}_f = \liminf_{r \to \infty} \frac{\log^3 M(r, f)}{\log r}.
\]

**Definition 3.** The type \( \sigma_f \) of a meromorphic function \( f \) is defined as
\[
\sigma_f = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
\]
When \( f \) is entire, then
\[
\sigma_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.
\]

**Definition 4.** A meromorphic function \( a = a(z) \) is called small with respect to \( f \) if \( T(r, a) = S(r, f) \).

**Definition 5.** Let \( a_1, a_2, \ldots, a_k \) be linearly independent meromorphic functions and small with respect to \( f \). We denote by \( L(f) = W(a_1, a_2, \ldots, a_k, f) \) the Wronskian determinant of \( a_1, a_2, \ldots, a_k, f \). i.e.
\[
L(f) = \begin{vmatrix}
  a_1 & a_2 & \ldots & a_k & f \\
  a_1' & a_2' & \ldots & a_k' & f' \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_1^{(k)} & a_2^{(k)} & \ldots & a_k^{(k)} & f^{(k)}
\end{vmatrix}
\]

**Definition 6.** If \( a \in \mathbb{C} \cup \{\infty\} \), the quantity
\[
\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}
\]
is called the Nevanlinna deficiency of the value \( a \).
From the second fundamental theorem it follows that the set of values of \( a \in \mathbb{C} \cup \{\infty\} \) for which \( \delta(a; f) > 0 \) is countable and
\[
\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) \leq 2.
\]
(cf.\([1, p.43]\)). If in particular,
\[
\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2,
\]
we say that \( f \) has the maximum deficiency sum.

Somasundaram and Thamizharasi \([3]\) introduced the notion of \( L - \text{order} \) and \( L - \text{type} \) for entire functions where \( L = L(r) \) is a positive continuous function increasing slowly in the sense of 'Karamata' i.e. \( L(ar) \sim L(r) \) as \( r \to \infty \) for every positive constant 'a'. Their definitions are as follows:

**Definition 7.** \([3]\) The \( L - \text{order} \) \( \rho^L_f \) and \( L - \text{lower order} \) \( \lambda^L_f \) of an entire function \( f \) are defined as follows:
\[
\rho^L_f = \limsup_{r \to \infty} \frac{\log[2] M(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda^L_f = \liminf_{r \to \infty} \frac{\log[2] M(r, f)}{\log [rL(r)]}.
\]
When \( f \) is meromorphic, then
\[
\rho^L_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda^L_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]}.
\]

**Definition 8.** \([3]\) The \( L - \text{type} \) \( \sigma^L_f \) of an entire function \( f \) with \( L\)-order \( \rho^L_f \) is defined as
\[
\sigma^L_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{[rL(r)]^{\rho^L_f}}, \quad 0 < \rho^L_f < \infty.
\]
For meromorphic \( f \), the \( L - \text{type} \) \( \sigma^L_f \) becomes
\[
\sigma^L_f = \limsup_{r \to \infty} \frac{T(r, f)}{[rL(r)]^{\rho^L_f}}, \quad 0 < \rho^L_f < \infty.
\]
Similarly one can define the \( L - \text{hyper order} \) and \( L - \text{hyper lower order} \) of entire and meromorphic \( f \).

The more generalised concept of \( L - \text{order} \) and \( L - \text{type} \) of entire and meromorphic functions are \( L^* - \text{order} \) and \( L^* - \text{type} \) respectively. Their definitions are as follows:
Definition 9. The $L^*$—order, $L^*$—lower order and $L^*$—type of a meromorphic function $f$ are defined by

$$
\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [re^{L(r)}]}, \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [re^{L(r)}]}
$$

and

$$
\sigma_f^{L^*} = \limsup_{r \to \infty} \frac{T(r, f)}{[re^{L(r)}]^\rho_f^{L^*}}, \quad 0 < \rho_f^{L^*} < \infty.
$$

When $f$ is entire, one can easily verify that

$$
\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log [2] M(r, f)}{\log [re^{L(r)}]}, \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log [2] M(r, f)}{\log [re^{L(r)}]}
$$

and

$$
\sigma_f^{L^*} = \limsup_{r \to \infty} \frac{\log M(r, f)}{[re^{L(r)}]^\rho_f^{L^*}}, \quad 0 < \rho_f^{L^*} < \infty,
$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper we establish the relationship between the $L$—order($L$—type) of a transcendental meromorphic function $f$ and that of a wronskian generated by it.

2 Lemma.

In this section we present a lemma which will be needed in the sequel.

Lemma 1. [2] Let $f$ be a transcendental meromorphic function having the maximum deficiency sum. Then

$$
\lim_{r \to \infty} \frac{T(r, L(f))}{T(r, f)} = 1 + k - k\delta(\infty; f).
$$

3 Theorems.

In this section we present the main results of the paper.

Theorem 1. Let $f$ be a transcendental meromorphic function having the maximum deficiency sum. Then the $L$—order of $L(f)$ are same as that of $f$. Also the $L$—type of $L(f)$ is $\{1 + k - k\delta(\infty; f)\}$ times that of $f$ if $f$ is of finite positive order.
Proof. By Lemma 1, \( \lim_{r \to \infty} \frac{\log T(r, L(f))}{\log T(r, f)} \) exists and is equal to 1. So

\[
\rho_{L(f)}^L = \limsup_{r \to \infty} \frac{\log T(r, L(f))}{\log [rL(r)]} = \limsup_{r \to \infty} \left\{ \frac{\log T(r, f) \log T(r, L(f))}{\log [rL(r)]} \right\} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]} \lim_{r \to \infty} \frac{\log T(r, f)}{\log T(r, L(f))} = \rho_f^L. 1 = \rho_f^L.
\]

Again

\[
\sigma_{L(f)}^L = \limsup_{r \to \infty} \frac{T(r, L(f))}{[rL(r)]^{\rho_{L(f)}^L}} = \limsup_{r \to \infty} \frac{T(r, f)}{[rL(r)]^{\rho_f^L}} \lim_{r \to \infty} \frac{T(r, L(f))}{T(r, f)} = \sigma_f^L. \{1 + k - k\delta(\infty; f)\}.
\]

This proves the theorem.

**Theorem 2.** Let \( f \) be a transcendental meromorphic function having the maximum deficiency sum. Then the \( L \)-lower order of \( L(f) \) and that of \( f \) are equal.

We omit the proof of Theorem 2 because it can be carried out in the line of Theorem 1.

**Theorem 3.** If \( f \) be a transcendental meromorphic function having the maximum deficiency sum then the \( L \)-hyper order of \( L(f) \) are same as that of \( f \).

Proof. Let \( \rho_{L(f)}^L \) and \( \rho_f^L \) be the \( L \)-hyper orders of \( L(f) \) and \( f \) respectively. By Lemma 1,

\[
\lim_{r \to \infty} \frac{\log^2 T(r, L(f))}{\log^2 T(r, f)}
\]


exists and is equal to 1. Thus we get

\[-L_{\rho_{L(f)}} = \limsup_{r \to \infty} \frac{\log^{[2]} T(r, L(f))}{\log [rL(r)]} = \limsup_{r \to \infty} \left\{ \frac{\log^{[2]} T(r, f)}{\log [rL(r)]}, \frac{\log^{[2]} T(r, L(f))}{\log [rL(r)]} \right\} = \limsup_{r \to \infty} \frac{\log^{[2]} T(r, f)}{\log [rL(r)]}, \lim_{r \to \infty} \frac{\log^{[2]} T(r, L(f))}{\log [rL(r)]} \]

\[-L_{\rho_{f.1}} = -L_{\rho_{f}}.

Thus the theorem is established.

In the line of Theorem 3 we may state the following theorem without proof.

**Theorem 4.** Let \( f \) be a transcendental meromorphic function having the maximum deficiency sum. Then the \( L - \) hyper lower orders of \( L(f) \) and \( f \) are same.

In the following theorem we establish the relationship between the \( L^* - \) order \((L^* - \text{type})\) of \( L(f) \) and \( f \).

**Theorem 5.** Let \( f \) be a transcendental meromorphic function having the maximum deficiency sum. Then the \( L^* - \) order of \( L(f) \) is same as that of \( f \). Also the \( L^* - \) type of \( L(f) \) is \( \{1 + k - k\delta(\infty; f)\} \) times that of \( f \) when \( f \) is of finite positive order.

**Proof.** By Lemma 1, \( \lim_{r \to \infty} \frac{\log T(r, L(f))}{\log T(r, f)} \) exists and is equal to 1. So

\[\rho^{L^*}_{L(f)} = \limsup_{r \to \infty} \frac{\log T(r, L(f))}{\log [r^{L^*(r)}]} = \limsup_{r \to \infty} \left\{ \frac{\log T(r, L(f))}{\log T(r, f)}, \frac{\log T(r, f)}{\log [r^{L^*(r)}]} \right\} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [r^{L^*(r)}]}, \lim_{r \to \infty} \frac{\log T(r, L(f))}{\log T(r, f)} \]

\[= \rho^{L^*}_{f.1} = \rho^{L^*}_{f}.


Again

\[ \sigma_{L(f)}^* = \limsup_{r \to \infty} \frac{T(r, L(f))}{[r e^{L(r)}]^\rho_{L(f)}} \]

\[ = \limsup_{r \to \infty} \left\{ \frac{T(r, f)}{[r e^{L(r)}]^\rho_{L(f)}} \cdot \frac{T(r, L(f))}{T(r, f)} \right\} \]

\[ = \limsup_{r \to \infty} \frac{T(r, f)}{[r e^{L(r)}]^\rho_{L(f)}} \cdot \lim_{r \to \infty} \frac{T(r, L(f))}{T(r, f)} \]

\[ = \sigma_f^* \cdot \{1 + k - k \delta(\infty; f)\}. \]

This proves the theorem.

**Theorem 6.** If \( f \) be a transcendental meromorphic function having the maximum deficiency sum then the \( L^* \) – lower order of \( L(f) \) and that of \( f \) are equal.

We omit the proof of Theorem 6 because it can be carried out in the line of Theorem 5.

**References**


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