A Note on the Maximum Terms of Composite Entire Functions

Sanjib Kumar Datta

Department of Mathematics, University of North Bengal
Darjeeling, Pin-734013, West Bengal, India
sanjib.kr_datta@yahoo.co.in

Sanjib Mondal

Chaltia Sreeguru Pathasala High School
P.O.-Berhampore, Dist.-Murshidabad, PIN-742101
West Bengal, India
sanjib_mondal_math@yahoo.in

Abstract
In the paper we compare the maximum term of composition of two entire functions with their corresponding left and right factors on the basis of $L - (p, q)^{th}$ order where $L = L(r)$ is a slowly changing function and $p, q$ are positive integers with $p > q$.

Mathematics Subject Classification: 30D30, 30D35

Keywords: Entire function, maximum term, composition, growth, slowly changing function, $L - (p, q)^{th}$ order, $L - (p, q)^{th}$ lower order, $L^{*} - (p, q)^{th}$ order, $L^{*} - (p, q)^{th}$ lower order, meromorphic function

1 Introduction, Definitions and Notations.

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r, f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r, f) = \max_{n \geq 0} (|a_n| r^n)$.

To start our paper we just recall the following definitions.

Definition 1. The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ is defined as follows:

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r, f)}{\log r}$$
where \( \log^k x = \log(\log^{k-1} x) \) for \( k = 1, 2, 3, \ldots \) and \( \log^0 x = x \).

**Definition 2.** The hyper order \( \overline{\rho}_f \) and hyper lower order \( \underline{\lambda}_f \) of \( f \) is defined by

\[
\overline{\rho}_f = \limsup_{r \to \infty} \frac{\log^3 M(r, f)}{\log r} \quad \text{and} \quad \underline{\lambda}_f = \liminf_{r \to \infty} \frac{\log^3 M(r, f)}{\log r}.
\]

Since for \( 0 \leq r < R \),

\[
\mu(r, f) \leq M(r, f) \leq \frac{R}{R - r} \mu(R, f),
\]

it is easy to see that

\[
\overline{\rho}_f = \limsup_{r \to \infty} \frac{\log^2 \mu(r, f)}{\log r}, \quad \underline{\lambda}_f = \liminf_{r \to \infty} \frac{\log^2 \mu(r, f)}{\log r}.
\]

and

\[
\overline{\rho}_f = \limsup_{r \to \infty} \frac{\log^3 \mu(r, f)}{\log r}, \quad \underline{\lambda}_f = \liminf_{r \to \infty} \frac{\log^3 \mu(r, f)}{\log r}.
\]

Somasundaram and Thamizharasi [3] introduced the notions of \( L \)-order, \( L \)-lower order and \( L \)-type for entire functions where \( L = L(r) \) is a positive continuous function increasing slowly in the sense of ‘Karamata’ i.e., \( L(ar) \sim L(r) \) as \( r \to \infty \) for every constant \( a \). Their definitions are as follows:

**Definition 3.** [3] The \( L \)-order \( \rho^L_f \) and \( L \)-lower order \( \lambda^L_f \) of an entire function \( f \) are defined as follows:

\[
\rho^L_f = \limsup_{r \to \infty} \frac{\log^2 M(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda^L_f = \liminf_{r \to \infty} \frac{\log^2 M(r, f)}{\log [rL(r)]}.
\]

When \( f \) is meromorphic, then

\[
\rho^L_f = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda^L_f = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]}.
\]

**Definition 4.** [3] The \( L \)-type \( \sigma^L_f \) of an entire function \( f \) with \( L \)-order \( \rho^L_f \) is defined as

\[
\sigma^L_f = \limsup_{r \to \infty} \frac{\log M(r, f)}{[rL(r)]^{\rho^L_f}}, \quad 0 < \rho^L_f < \infty.
\]

For meromorphic \( f \), the \( L \)-type \( \sigma^L_f \) becomes

\[
\sigma^L_f = \limsup_{r \to \infty} \frac{T(r, f)}{[rL(r)]^{\rho^L_f}}, \quad 0 < \rho^L_f < \infty.
\]
With the help of the notion of maximum terms of entire functions, Definition 3 and Definition 4 can be alternatively stated as follows:

**Definition 5.** The $L$–order $\rho_f^L$ and the $L$–lower order $\lambda_f^L$ of an entire function $f$ are defined as follows:

$$
\rho_f^L = \limsup_{r \to \infty} \frac{\log[2] \mu(r, f)}{\log |rL(r)|} \quad \text{and} \quad \lambda_f^L = \liminf_{r \to \infty} \frac{\log[2] \mu(r, f)}{\log |rL(r)|}.
$$

When $f$ is meromorphic, then $\rho_f^L$ and $\lambda_f^L$ cannot be defined in the above way.

**Definition 6.** The $L$–type $\sigma_f^L$ of an entire function $f$ with $L$–order $\rho_f^L$ is defined as

$$
\sigma_f^L = \limsup_{r \to \infty} \frac{\log \mu(r, f)}{[rL(r)]^{\rho_f^L}}, \quad 0 < \rho_f^L < \infty.
$$

For meromorphic $f$, the $L$–type $\sigma_f^L$ cannot be defined in the above way.

Juneja, Kapoor and Bajpai[2] defined the $(p, q)$th order and $(p, q)$th lower order of an entire function $f$ respectively as follows:

$$
\rho_f(p, q) = \limsup_{r \to \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} r}
$$

and

$$
\lambda_f(p, q) = \liminf_{r \to \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} r}.
$$

When $f$ is meromorphic, one can easily verify that

$$
\rho_f(p, q) = \limsup_{r \to \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} r}
$$

and

$$
\lambda_f(p, q) = \liminf_{r \to \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} r},
$$

where $p, q$ are positive integers and $p > q$.

With the notion of slowly changing function one can easily define the following:
Definition 7. The \( L - (p, q) \)th order and \( L - (p, q) \)th lower order of an entire function \( f \) are respectively defined as:

\[
\rho_f^{(p, q)} = \limsup_{r \to \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} [rL(r)]},
\]

and

\[
\lambda_f^{(p, q)} = \liminf_{r \to \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} [rL(r)]}.
\]

When \( f \) is meromorphic one can easily verify that

\[
\rho_f(p, q) = \limsup_{r \to \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} [rL(r)]},
\]

and

\[
\lambda_f(p, q) = \liminf_{r \to \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} [rL(r)]},
\]

where \( p, q \) are positive integers and \( p > q \).

In view of the notion of maximum terms of entire functions, Definition 7 can be restated in the following way:

Definition 8. The \( L - (p, q) \)th order and \( L - (p, q) \)th lower order of an entire function \( f \) are respectively defined as:

\[
\rho_f^{L}(p, q) = \limsup_{r \to \infty} \frac{\log^{[p+1]} \mu(r, f)}{\log^{[q]} [rL(r)]},
\]

and

\[
\lambda_f^{L}(p, q) = \liminf_{r \to \infty} \frac{\log^{[p+1]} \mu(r, f)}{\log^{[q]} [rL(r)]},
\]

where \( p, q \) are positive integers and \( p > q \).

When \( f \) is meromorphic, then \( \rho_f^{L}(p, q) \) and \( \lambda_f^{L}(p, q) \) cannot be defined in the above way.

The more generalised concept of \( L - \)order and \( L - \)type of entire and meromorphic functions are \( L^* - \)order and \( L^* - \)type respectively. Their definitions are as follows:

Definition 9. The \( L^* - \)order, \( L^* - \)lower order and \( L^* - \)type of a meromorphic function are defined by

\[
\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [rE(r)]}, \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [rE(r)]}
\]

for meromorphic functions.
and
\[
\sigma_f^{L^*} = \limsup_{r \to \infty} \frac{T(r, f)}{[r e^{L(r)}]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.
\]

When \( f \) is entire, one can easily verify that
\[
\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log M(r, f)}{\log [r e^{L(r)}]}, \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log M(r, f)}{\log [r e^{L(r)}]},
\]
and
\[
\sigma_f^{L^*} = \limsup_{r \to \infty} \frac{\log M(r, f)}{[r e^{L(r)}]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.
\]

In view of the notion of maximum terms of entire functions we may state the following definition.

**Definition 10.** The \( L^* - (p, q) \)th order and \( L^* - (p, q) \)th lower order of an entire function \( f \) are respectively defined as:

\[
\rho_f^{L^*}(p, q) = \limsup_{r \to \infty} \frac{\log^{[p+1]} \mu(r, f)}{\log^{[q]} [r e^{L(r)}]},
\]
and
\[
\lambda_f^{L^*}(p, q) = \liminf_{r \to \infty} \frac{\log^{[p+1]} \mu(r, f)}{\log^{[q]} [r e^{L(r)}]},
\]

where \( p, q \) are positive integers and \( p > q \).

When \( f \) is meromorphic, then \( \rho_f^{L^*}(p, q) \) and \( \lambda_f^{L^*}(p, q) \) cannot be defined in the above way.

Singh [4] proved some theorems on the comparative growth properties of \( \log^{[2]} \mu(r, f \circ g) \) with respect to \( \log^{[2]} \mu(r^A, f) \) for every positive constant \( A \). In the paper we further investigate the comparative growths of maximum term of two entire functions with their corresponding left and right factors on the basis of \( L - (p, q) \)th order and \( L - (p, q) \)th lower order where \( p, q \) are positive integers and \( p > q \). We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [5] and [1].

**2 Theorems:**

In this section we present the main results of the paper.
Theorem 1. Let $f$ and $g$ be two entire functions such that $0 < \lambda^L_{fg}(p, q) \leq \rho^L_{fg}(p, q) < \infty$ and $0 < \rho^L_g(m, q) < \infty$ where $p, q, m$ are positive integers such that $q < \min \{p, m\}$. Then for any integer $A$

\[
(i) \liminf_{r \to \infty} \frac{\log^{|p|}{\mu(r, f \circ g)}}{\log^{|m|}{\mu{(r^A, g)}}} \leq \frac{\rho^L_{fg}(p, q)}{A \rho^L_g(m, q)} \leq \limsup_{r \to \infty} \frac{\log^{|p|}{\mu(r, f \circ g)}}{\log^{|m|}{\mu{(r^A, g)}}}.
\]

Further if $\lambda^L_g(m, q) > 0$ then

\[
(ii) \quad \frac{\lambda^L_{fg}(p, q)}{A \rho^L_g(m, q)} \leq \liminf_{r \to \infty} \frac{\log^{|p|}{\mu(r, f \circ g)}}{\log^{|m|}{\mu{(r^A, g)}}} \leq \frac{\lambda^L_{fg}(p, q)}{A \lambda^L_g(m, q)} \leq \frac{\rho^L_{fg}(p, q)}{A \rho^L_g(m, q)}.
\]

and

\[
(iii) \quad \liminf_{r \to \infty} \frac{\log^{|p|}{\mu(r, f \circ g)}}{\log^{|m|}{\mu{(r^A, g)}}} \leq \min \left\{ \frac{\lambda^L_{fg}(p, q)}{A \lambda^L_g(m, q)}, \frac{\rho^L_{fg}(p, q)}{A \rho^L_g(m, q)} \right\}
\]

\[
\leq \max \left\{ \frac{\lambda^L_{fg}(p, q)}{A \lambda^L_g(m, q)}, \frac{\rho^L_{fg}(p, q)}{A \rho^L_g(m, q)} \right\} = \limsup_{r \to \infty} \frac{\log^{|p|}{\mu(r, f \circ g)}}{\log^{|m|}{\mu{(r^A, g)}}}.
\]

Proof. (i) From the definition of $L - (p, q)$th order we have for arbitrary positive $\epsilon$ and for all large values of $r$,

\[
\log^{|p|}{\mu(r, f \circ g)} \leq (\rho^L_{fg}(p, q) + \epsilon) \log^{|q|}{rL(r)}
\]  

(1)

and for a sequence of values of $r$ tending to infinity,

\[
\log^{|m|}{\mu{(r^A, g)}} \geq A (\rho^L_g(m, q) - \epsilon) \log^{|q|}{rL(r)}.
\]  

(2)

Now from (1) and (2) it follows for a sequence of values of $r$ tending to infinity,

\[
\frac{\log^{|p|}{\mu(r, f \circ g)}}{\log^{|m|}{\mu{(r^A, g)}}} \leq \frac{\rho^L_{fg}(p, q) + \epsilon}{A (\rho^L_g(m, q) - \epsilon)}.
\]

As $\epsilon (> 0)$ is arbitrary we obtain that

\[
\liminf_{r \to \infty} \frac{\log^{|p|}{\mu(r, f \circ g)}}{\log^{|m|}{\mu{(r^A, g)}}} \leq \frac{\rho^L_{fg}(p, q)}{A \rho^L_g(m, q)}.
\]  

(3)

Again for a sequence of values of $r$ tending to infinity,

\[
\log^{|p|}{\mu(r, f \circ g)} \geq (\rho^L_{fg}(p, q) - \epsilon) \log^{|q|}{rL(r)}.
\]  

(4)
Also for all sufficiently large values of \( r \),
\[
\log^{[m]} \mu(r^A, g) \leq A \left( \rho^L_g(m, q) + \epsilon \right) \log^{[q]} [rL(r)].
\] (5)

So combining (4) and (5) we get for a sequence of values of \( r \) tending to infinity,
\[
\frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \geq \frac{\rho^L_{f \circ g}(p, q) - \epsilon}{A(\rho^L_g(m, q) + \epsilon)}.
\]

Since \( \epsilon (> 0) \) is arbitrary it follows that
\[
\limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \geq \frac{\rho^L_{f \circ g}(p, q)}{A\rho^L_g(m, q)}.
\] (6)

Thus (i) follows from (3) and (6).

(ii) From the definition of \( L - (p, q) \)th lower order we have for arbitrary positive \( \epsilon \) and for all large values of \( r \),
\[
\log^{[p]} \mu(r, f \circ g) \geq \left( \lambda^L_{f \circ g}(p, q) - \epsilon \right) \log^{[q]} [rL(r)].
\] (7)

Now from (5) and (7) it follows for all large values of \( r \),
\[
\frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \geq \frac{\lambda^L_{f \circ g}(p, q) - \epsilon}{A(\rho^L_g(m, q) + \epsilon)}.
\]

As \( \epsilon (> 0) \) is arbitrary we obtain that
\[
\liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \geq \frac{\lambda^L_{f \circ g}(p, q)}{A\lambda^L_g(m, q)}.
\] (8)

Again for a sequence of values of \( r \) tending to infinity,
\[
\log^{[p]} \mu(r, f \circ g) \leq \left( \lambda^L_{f \circ g}(p, q) + \epsilon \right) \log^{[q]} [rL(r)]
\]
(9)

and for all large values of \( r \),
\[
\log^{[m]} \mu(r^A, g) \geq A \left( \lambda^L_g(m, q) - \epsilon \right) \log^{[q]} [rL(r)].
\] (10)

So combining (9) and (10) we get for a sequence of values of \( r \) tending to infinity,
\[
\frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \leq \frac{\lambda^L_{f \circ g}(p, q) + \epsilon}{A(\lambda^L_g(m, q) - \epsilon)}.
\]

Since \( \epsilon (> 0) \) is arbitrary it follows that
\[
\liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \leq \frac{\lambda^L_{f \circ g}(p, q)}{A\lambda^L_g(m, q)}.
\] (11)
Also for a sequence of values of \( r \) tending to infinity,
\[
\log^m \mu(r^A, g) \leq A \left( \lambda^L_g(m, q) + \epsilon \right) \log^q [rL(r)].
\]
(12)

Now from (7) and (12) we obtain for a sequence of values of \( r \) tending to infinity,
\[
\log^m \mu(r, f \circ g) \leq \frac{\lambda^L_{f \circ g}(p, q) - \epsilon}{A(\lambda^L_g(m, q) + \epsilon)}.
\]

As \( \epsilon > 0 \) is arbitrary we get that
\[
\limsup_{r \to \infty} \log^m \mu(r, f \circ g) \geq \frac{\lambda^L_{f \circ g}(p, q)}{A(\lambda^L_g(m, q) + \epsilon)}.
\]
(13)

Again from (1) and (10) it follows for all large values of \( r \),
\[
\log^m \mu(r, f \circ g) \leq \frac{\rho^L_{f \circ g}(p, q) + \epsilon}{A(\lambda^L_g(m, q) - \epsilon)}.
\]

As \( \epsilon > 0 \) is arbitrary we obtain that
\[
\limsup_{r \to \infty} \log^m \mu(r, f \circ g) \leq \frac{\rho^L_{f \circ g}(p, q)}{A(\lambda^L_g(m, q) + \epsilon)}.
\]
(14)

Thus (ii) follows from (8),(11),(13) and (14).

(iii) Combining (i) and (ii) of Theorem 1, (iii) follows.

**Theorem 2.** If \( f \) and \( g \) be two entire functions with \( \rho^L_g(m, q) < \infty \) and \( \rho^L_{f \circ g}(p, q) = \infty \), then for every positive number \( A \),
\[
\limsup_{r \to \infty} \frac{\log^m \mu(r, f \circ g)}{\log^m \mu(r^A, g)} = \infty,
\]
where \( p, q, m \) are positive integers with \( q < \min \{ p, m \} \).

**Proof.** Let us assume that the conclusion of Theorem 2 does not hold. Then there exists a constant \( B > 0 \) such that for all sufficiently large values of \( r \),
\[
\log^m \mu(r, f \circ g) \leq B \log^m \mu(r^A, g).
\]
(15)

Again from the definition of \( \rho^L_g(m, q) \) it follows that
\[
\log^m \mu(r^A, g) \leq (\rho^L_g(m, q) + \epsilon)A \log^q [rL(r)]
\]
(16)
holds for all large values of \( r \). So from (15) and (16) we obtain for all sufficiently large values of \( r \),
\[
\log^m \mu(r, f \circ g) \leq (\rho^L_g(m, q) + \epsilon)AB \log^q [rL(r)].
\]
(17)

From (17) it follows that \( \rho^L_{f \circ g}(p, q) < \infty \).

So we arrive at a contradiction. This proves the theorem.
Remark 1. If we take $\rho^*_f(p, q) < \infty$ instead of $\rho^*_g(m, q) < \infty$ in Theorem 2 and the other conditions remain the same then the theorem remains valid with $g$ replaced by $f$ in the denominator.

In the line of Theorem 1 and Theorem 2 we may respectively state the following two theorems without proof.

Theorem 3. Let $f$ and $g$ be two entire functions such that $0 < \lambda^*_f(p, q) \leq \rho^*_f(p, q) < \infty$ and $0 < \lambda^*_g(m, q) < \infty$ where $p, q, m$ are positive integers such that $q < \min\{p, m\}$. Then for any integer $A$

\[
\begin{align*}
(i) \quad & \liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \leq \frac{\rho^*_f(p, q)}{\lambda^*_g(m, q)} \leq \limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)}. \\
(ii) \quad & \frac{\lambda^*_f(p, q)}{A \lambda^*_g(m, q)} \leq \liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \leq \frac{\lambda^*_f(p, q)}{A \lambda^*_g(m, q)} \leq \limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \\
(iii) \quad & \liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \leq \min \left\{ \frac{\lambda^*_f(p, q)}{A \lambda^*_g(m, q)}, \frac{\rho^*_f(p, q)}{A \rho^*_g(m, q)} \right\} \\
& \leq \max \left\{ \frac{\lambda^*_f(p, q)}{A \lambda^*_g(m, q)}, \frac{\rho^*_f(p, q)}{A \rho^*_g(m, q)} \right\} \leq \limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)}. 
\end{align*}
\]

Further if $\lambda^*_g(m, q) > 0$ then

\[
\begin{align*}
(ii) \quad & \frac{\lambda^*_f(p, q)}{A \lambda^*_g(m, q)} \leq \liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \leq \frac{\lambda^*_f(p, q)}{A \lambda^*_g(m, q)} \leq \limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \\
(iii) \quad & \liminf_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} \leq \min \left\{ \frac{\lambda^*_f(p, q)}{A \lambda^*_g(m, q)}, \frac{\rho^*_f(p, q)}{A \rho^*_g(m, q)} \right\} \\
& \leq \max \left\{ \frac{\lambda^*_f(p, q)}{A \lambda^*_g(m, q)}, \frac{\rho^*_f(p, q)}{A \rho^*_g(m, q)} \right\} \leq \limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)}. 
\end{align*}
\]

Theorem 4. If $f$ and $g$ be two entire functions with $\rho^*_g(p, q) < \infty$ and $\rho^*_f(p, q) = \infty$, then for every positive number $A$,

\[
\limsup_{r \to \infty} \frac{\log^{[p]} \mu(r, f \circ g)}{\log^{[m]} \mu(r^A, g)} = \infty,
\]

where $p, q, m$ are positive integers with $q < \min\{p, m\}$.

References


Received: April, 2009