On the Spherical Representatives of a Curve

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Abstract

In this work we study the spherical indicatrices (images) of Frenet vector fields of a curve which lies on a hypersurface in n-dimensional Euclidean space. We investigate the relation of spherical involute and Bertrand mate for these indicatrix curves.

As a result, we obtain spherical indicatrices which construct spherical involute and Bertrand mates with each other.

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1 Introduction

In [1], spherical indicatrices, evolutes and involutes of curves were given. Bertrand curves and their characterizations were investigated in [2].

In this paper we investigate evolutes, involutes and Bertrand curves among spherical indicatrix curves. We take a curve on a hypersurface in $E^n$ for $n \geq 3$ which has (n-1) pieces of Frenet vectors at each point. The end points of Frenet vector fields form spherical indicatrix curves by moving them to the centre of the sphere [1]. In section 3, we search spherical involutes of each spherical indicatrix through other spherical indicatrices of Frenet vector fields. We make a generalization of spherical involutes in a hypersurface of $E^n$. In section 4, we investigate the relation of Bertrand mates of spherical indicatrices. We reveal that which couples of spherical indicatrices of Frenet vector fields constitute Bertrand mate and also we make a generalization.
2 Basic notations and properties

We now recall some basic concepts on classical differential geometry of curves which lie on a hypersurface in $\mathbb{E}^n$. $\mathbb{E}^n$ is an n-dimensional Euclidean space and M is a hypersurface of $\mathbb{E}^n$. For any vectors $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$, we denote $\langle x, y \rangle$ as the standard inner product. Let $\alpha : I \rightarrow M \subset \mathbb{E}^n$ be a curve with unit speed on M and $s$ is arc-length parameter of the curve. Let the Frenet vector fields of the curve be $\{V_1(s), V_2(s), ..., V_{n-1}(s)\}$ and $i^{th}$ curvature at the point $s$, be $k_i(s); 1 \leq i \leq n - 1$. The spherical indicatrix curves are formed on unit sphere when Frenet vector fields are transported to the centre of the sphere [1]. As a notation, spherical indicatrices are denoted by $(V_i(s)); 1 \leq i \leq n - 1$. Let us indicate $V_1(s) = \alpha'(s)$, where $V_1(s)$ is a unit tangent vector of $\alpha$ at $s$. The unit principal normal vector $V_2(s)$ at $s$ is given by $V_2(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ when $s$ is arc-length parameter.

For $\{V_1(s), V_2(s), ..., V_{n-1}(s)\}$ Frenet frame, the following Frenet formula hold:

\[
\begin{align*}
V_1'(s) & = k_1(s)V_2(s) \\
V_i'(s) & = -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s); 1 < i < n - 1 \\
V_{n-1}'(s) & = -k_{n-2}(s)V_{n-2}(s)
\end{align*}
\]

A curve whose tangent vectors at each point are perpendicular to tangent vectors of another curve is called involute of the other curve. As it is expressed in [3], tangent indicatrix and binormal indicatrix of a curve in $\mathbb{E}^3$ are spherical involutes of constant pole curve.

We now give the statement of Bertrand curves. Let $\alpha$ and $\beta$ be curves with arc-length parameter, $s$, on a hypersurface in $\mathbb{E}^n$. $\{V_1(s), V_2(s), ..., V_{n-1}(s)\}$ and $\{V_1^*(s), V_2^*(s), ..., V_{n-1}^*(s)\}$ are Frenet vector fields of $\alpha$ and $\beta$, respectively. If $V_2(s)$ and $V_2^*(s)$ are linearly dependent then $\alpha$ and $\beta$ are called Bertrand mates. As a notation, $V_i(s)$ and $V_i^*(s); 1 \leq i \leq n - 1$ will denote tangent vector field and principal normal vector field of spherical indicatrices, respectively.

3 Spherical involutes of spherical indicatrices

In this section we investigate the spherical involutes of each spherical indicatrix.

We determine a curve, $\alpha : I \rightarrow M \subset \mathbb{E}^n$ with unit speed and arc-length parameter, $s$. The equation of spherical indicatrices are given by
\[ \alpha_i : I \longrightarrow S^{n-1} \]
\[ s \longrightarrow \alpha_i(s) = V_i(s); \quad 1 \leq i \leq n - 1 \]

\( S^{n-1} \) is a unit sphere and each of the spherical indicatrices are formed on this sphere. We investigate tangent vectors of spherical indicatrices which are perpendicular to each other to find relation of spherical involutes.

**Theorem 3.1** All spherical indicatrices which are denoted by \((V_j)\) are spherical involutes for \((V_j)\), except \((V_{j-2})\) and \((V_{j+2})\); \(1 \leq j \leq n - 1\). If \(j = 1, 2, (n - 2), (n - 1)\) then the number of spherical involutes are \((n - 3)\), otherwise they are \((n - 4)\).

**Proof.** For \(1 \leq j \leq n - 1\):
\[ \langle V'_j(s), V'_{j-4}(s) \rangle = \langle -k_{j-1}(s)V_{j-1}(s) + k_j(s)V_{j+1}(s), -k_{j-5}(s)V_{j-5}(s) + k_{j-4}(s)V_{j-3}(s) \rangle \]
\[ = 0 \]
\[ \langle V'_j(s), V'_{j-3}(s) \rangle = \langle -k_{j-1}(s)V_{j-1}(s) + k_j(s)V_{j+1}(s), -k_{j-4}(s)V_{j-4}(s) + k_{j-3}(s)V_{j-2}(s) \rangle \]
\[ = 0 \]
\[ \langle V'_j(s), V'_{j-2}(s) \rangle = \langle -k_{j-1}(s)V_{j-1}(s) + k_j(s)V_{j+1}(s), -k_{j-3}(s)V_{j-3}(s) + k_{j-2}(s)V_{j-1}(s) \rangle \]
\[ \neq 0 \]
\[ \langle V'_j(s), V'_{j+1}(s) \rangle = \langle -k_{j-1}(s)V_{j-1}(s) + k_j(s)V_{j+1}(s), -k_{j}(s)V_j(s) + k_{j+1}(s)V_{j+2}(s) \rangle \]
\[ = 0 \]
\[ \langle V'_j(s), V'_{j+2}(s) \rangle = \langle -k_{j-1}(s)V_{j-1}(s) + k_j(s)V_{j+1}(s), -k_{j+1}(s)V_{j+1}(s) + k_{j+2}(s)V_{j+3}(s) \rangle \]
\[ \neq 0 \]
\[ \langle V'_j(s), V'_{j+3}(s) \rangle = \langle -k_{j-1}(s)V_{j-1}(s) + k_j(s)V_{j+1}(s), -k_{j+2}(s)V_{j+2}(s) + k_{j+3}(s)V_{j+4}(s) \rangle \]
\[ = 0 \]
\[ \langle V'_j(s), V'_{j+4}(s) \rangle = \langle -k_{j-1}(s)V_{j-1}(s) + k_j(s)V_{j+1}(s), -k_{j+3}(s)V_{j+3}(s) + k_{j+4}(s)V_{j+5}(s) \rangle \]
\[ = 0 \]

So \((V_{j-2}(s))\) and \((V_{j+2}(s))\) are not spherical involutes of \((V_j(s))\), for \(1 \leq j \leq n-1\). \(\square\)
4 Bertrand mates of spherical indicatrices

For two curves that lie on a hypersurface in $E^n$, if principal normal vectors of these curves are linearly dependent then they are Bertrand mates. Let these curves be $\alpha$, $\beta$ with arc-length parameter $s$ and principal normal vectors at $s$ are $V_2(s)$, $V_2^*(s)$, respectively. Since $V_2(s)$ and $V_2^*(s)$ are linearly dependent, we can write:

$$V_2(s) = \lambda V_2^*(s)$$

where $\lambda$ is a constant. \{${V_1(s), V_2(s), ..., V_{n-1}(s)}$\} , \{${V_1^*(s), V_2^*(s), ..., V_{n-1}^*(s)}$\} are Frenet frame of $\alpha$ and $\beta$. Consequently, the following equalities are obtained:

$$\langle V_1(s), V_2^*(s) \rangle = 0$$
$$\langle V_1^*(s), V_2(s) \rangle = 0$$

4.1 Tangent vectors of spherical indicatrices

The tangent vector of $\alpha$, with arc-length parameter, $s$, is computed by;

$$V_i(s) = \alpha'(s)$$

We now calculate tangent vectors of each spherical indicatrix and denote them $V_{i1}(s)$, as a notation; $1 \leq i \leq n - 1$.

$$V_{11}(s) = k_1(s)V_2(s)$$
$$V_{i1}(s) = -k_{i-1}(s)V_{i-1}(s) + k_i(s)V_{i+1}(s) ; \quad 2 \leq i \leq (n-2)$$
$$V_{(n-1)1}(s) = -k_{n-2}(s)V_{n-2}(s)$$

4.2 Principal normal vectors of spherical indicatrices

The principal normal vector of $\alpha$, with arc-length parameter, $s$, is computed by;

$$V_2(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$$

We now calculate principal normal vectors of each spherical indicatrix and denote them $V_{i2}(s)$, as a notation; $1 \leq i \leq n - 1$.

$$V_{i2}(s) = \frac{-k_{i1}^2(s)V_1(s) + k_{i1}^2(s)V_2(s) + k_1(s)k_2(s)V_3(s)}{\sqrt{k_1^4(s) + k_1^2(s)^2 + k_1^2(s)k_2^2(s)}}$$
\[ V_{2s}(s) = \frac{-k'_1(s)V_1(s) - (k_1^2(s) + k_2^2(s))V_2(s) + k'_2(s)V_3(s) + k_2(s)k_3(s)V_4(s)}{\sqrt{k'_1(s)^2 + (k_1^2(s) + k_2^2(s))^2 + k'_2(s)^2 + k_2^2(s)k_3^2(s)}} \]

\[ V_{iz}(s) = \frac{k_{i-2}(s)k_{i-1}(s)V_{i-2}(s) - k'_{i-1}(s)V_{i-1}(s) - (k_{i-1}^2(s) + k_2^2(s))V_i(s) + k'_i(s)V_{i+1}(s) + k_i(s)k_{i+1}(s)V_{i+2}(s)}{\sqrt{k^2_{i-2}(s)k_{i-1}^2(s) + k_{i-1}^2(s)^2 + (k_{i-1}^2(s) + k_2^2(s))^2 + k'_i(s)^2 + k_2^2(s)k_{i+1}^2(s)}} \]

\[ 3 \leq i \leq (n - 3) \]

\[ V_{(n-2)_z}(s) = \frac{k_{n-4}(s)k_{n-3}(s)V_{n-4}(s) - k'_{n-3}(s)V_{n-3}(s) - (k_{n-3}^2(s) + k_{n-2}^2(s))V_{n-2}(s) + k'_n(s)V_{n-1}(s)}{\sqrt{k^2_{n-4}(s)k_{n-3}^2(s) + k_{n-3}^2(s)^2 + (k_{n-3}^2(s) + k_{n-2}^2(s))^2 + k'_n(s)^2 + k_{n-2}^2(s)^2}} \]

\[ V_{(n-1)_z}(s) = \frac{k_{n-3}(s)k_{n-2}(s)V_{n-3}(s) - k'_{n-2}(s)V_{n-2}(s) - k_{n-2}^2(s)V_{n-1}(s)}{\sqrt{k^2_{n-3}(s)k_{n-2}^2(s) + k_{n-2}^2(s)^2 + k^4_{n-2}(s)}} \]

**Theorem 4.1** All spherical indicatrices which are denoted by \((V_j)\) form Bertrand mate with \((V_j)\), except \((V_{j-3}), (V_{j-2}), (V_{j-1}), (V_{j+1})\), \((V_{j+2})\) and \((V_{j+3})\); \(1 \leq j \leq n - 1\). If \( j = 1, (n - 1) \) then the number of Bertrand mates are \((n - 5)\), if \( j = 2, (n - 2) \) then they are \((n - 6)\), if \( j = 3, (n - 3) \) then they are \((n - 7)\), otherwise they are \((n - 8)\).

**Proof.** For \(1 \leq j \leq n - 1\):

\[ \langle V_{j_1}(s), V_{j-4}(s) \rangle = 0 \quad \langle V_{j_2}(s), V_{j-4}(s) \rangle = 0 \]
\[ \langle V_{j_1}(s), V_{j-3}(s) \rangle \neq 0 \quad \langle V_{j_2}(s), V_{j-3}(s) \rangle \neq 0 \]
\[ \langle V_{j_1}(s), V_{j-2}(s) \rangle \neq 0 \quad \langle V_{j_2}(s), V_{j-2}(s) \rangle \neq 0 \]
\[ \langle V_{j_1}(s), V_{j-1}(s) \rangle \neq 0 \quad \langle V_{j_2}(s), V_{j-1}(s) \rangle \neq 0 \]
\[ \langle V_{j_1}(s), V_{j+1}(s) \rangle \neq 0 \quad \langle V_{j_2}(s), V_{j+1}(s) \rangle \neq 0 \]
\[ \langle V_{j_1}(s), V_{j+2}(s) \rangle \neq 0 \quad \langle V_{j_2}(s), V_{j+2}(s) \rangle \neq 0 \]
\[ \langle V_{j_1}(s), V_{j+3}(s) \rangle \neq 0 \quad \langle V_{j_2}(s), V_{j+3}(s) \rangle \neq 0 \]
\[ \langle V_{j_1}(s), V_{j+4}(s) \rangle = 0 \quad \langle V_{j_2}(s), V_{j+4}(s) \rangle = 0 \]

So \((V_{j-3}), (V_{j-2}), (V_{j-1}), (V_{j+1}), (V_{j+2})\) and \((V_{j+3})\) are not Bertrand mate for \((V_j)\); \(1 \leq j \leq n - 1\)
References


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