Some Equivalent Conditions on s-Normal Matrices

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Abstract

Concept of s-unitarily equivalent matrices is introduced. Some equivalent conditions on s-normal matrices are given.

Mathematics Subject Classifications: 15A09, 15A57

Keywords: Conjugate secondary transpose, secondary normal, secondary hermitian, eigenvalue, eigenvector

1. Introduction:

The concept of s-normal matrices was introduced in [1]. Equivalent conditions on normal matrices were given in [2]. In this paper, our intention is to define s-unitarily equivalent matrices and prove some equivalent conditions on s-normal matrices. Also we prove some results on s-normal matrices. Let $\mathcal{E}_{n \times n}$ be the space of $n \times n$ complex matrices. If $A = (a_{ij}) \in \mathcal{E}_{n \times n}$, then the secondary transpose of $A$, denoted by $A^s$, is defined as $A^s = (b_{ij})$, where $b_{ij} = a_{n-j+1,n-i+1}$. $A^s$ denotes the conjugate secondary transpose of $A$, i.e. $A^s = \overline{A^s} = (c_{ij})$, where $c_{ij} = \overline{a_{n-j+1,n-i+1}}$. As $A^T$ satisfies the elementary properties, $A^s$ also satisfies the
properties such as \((A^\theta)^\theta = A\), \((A + B)^\theta = A^\theta + B^\theta\), \((AB)^\theta = B^\theta A^\theta\) etc., where \(A, B \in \mathbb{C}^{n \times n}\). A matrix \(A \in \mathbb{C}^{n \times n}\) is said to be s-hermitian matrix if \(A^\theta = A\).

2. Definitions:

In this section, we define s-normal, s-unitary and s-unitarily equivalent matrices.

**Definition 2.1:**

A matrix \(A \in \mathbb{C}^{n \times n}\) is said to be secondary normal (s-normal) if \(AA^\theta = A^\theta A\).

**Example 2.2:**

\[
A = \begin{pmatrix}
6 + 2i & 3 \\
2 & 6 + 4i
\end{pmatrix}
\]

is an s-normal matrix.

**Definition 2.3:**

A matrix \(A \in \mathbb{C}^{n \times n}\) is said to be secondary unitary (s-unitary) if \(AA^\theta = A^\theta A = I\).

**Example 2.4:**

\[
A = \begin{pmatrix}
i & 1 \\
\frac{\sqrt{2}}{i} & \frac{\sqrt{2}}{i}
\end{pmatrix}
\]

is an s-unitary matrix.

**Definition 2.5:**

Let \(A, B \in \mathbb{C}^{n \times n}\). The matrix \(B\) is said to be secondary unitarily equivalent (s-unitarily equivalent) to \(A\) if there exists an s-unitary matrix \(U\) such that \(B = U^\theta AU\).

**Example 2.6:**

Let \(A = \begin{pmatrix} 1 + i & 2i \\ 3 + 2i & 3 \end{pmatrix}\) and \(B = \begin{pmatrix} 2 + 2i & 2 + 3i \\ -2 + 2i & -3 + 2i \end{pmatrix}\).
Then if we take \( U = \begin{pmatrix} \frac{i}{\sqrt{2}} & 1 \\ 1 & \frac{i}{\sqrt{2}} \end{pmatrix} \), it can be verified that \( UU^0 = U^0U = I \) and \( B = U^0AU \). Hence \( B \) is s-unitarily equivalent to \( A \).

3. Equivalent conditions on s-normal matrices:

**Theorem 3.1:**

Let \( A \in \mathcal{C}_{n \times n} \). If \( A \) is s-unitarily equivalent to a diagonal matrix, then \( A \) is s-normal.

**Proof:**

Let \( A \in \mathcal{C}_{n \times n} \). If \( A \) is s-unitarily equivalent to a diagonal matrix \( D \), then there exists an s-unitary matrix \( P \) such that \( P^0AP = D \) which implies that \( A = PD^0P \), as \( P^0P = I \). Now \( AA^0 = PD^0P^0D^0P = PDD^0P^0 \). Also \( A^0A = PD^0P^0D^0P = PD^0D^0 \). Since \( D \) and \( D^0 \) are each diagonal, \( DD^0 = D^0D \) and hence \( AA^0 = A^0A \) so that \( A \) is s-normal.

**Remark 3.2:** It can be shown that \( A \) is s-normal \( \iff A^{-1}A^0 \) is s-unitary.

**Theorem 3.3:**

Let \( H, N \in \mathcal{C}_{n \times n} \) be invertible. If \( B = HNH \), where \( H \) is s-hermitian and \( N \) is s-normal, then \( B^{-1}B^0 \) is similar to an s-unitary matrix.

**Proof:**

Let \( H, N \in \mathcal{C}_{n \times n} \) be invertible. If \( B = HNH \), then \( B^{-1}B^0 = H^{-1}N^{-1}H^0N^0H^0 = H^{-1}N^{-1}H^{-1}HN^0H \) as \( H^0 = H \) and hence \( B^{-1}B^0 = H^{-1}N^{-1}N^0H \). Since \( N \) is s-normal, from remark 3.2, \( N^{-1}N^0 \) is s-unitary and hence the result follows.
Theorem 3.4: If $A$ is s-normal and $AB = 0$, then $A^\theta B = 0$.

Proof: See [1].

Theorem 3.5: If $X$ is an eigenvector of an s-normal matrix $A$ corresponding to an eigenvalue $\lambda$, then $X$ is also an eigenvector of $A^\theta$ corresponding to the eigenvalue $\overline{\lambda}$.

Proof: Let $A \in \mathcal{E}_{n \times n}$ be s-normal. Since $X$ is an eigenvector of $A$ corresponding to an eigenvalue $\lambda$, $AX = \lambda X$. Since $A$ is s-normal, it can be easily seen that $A-\lambda I$ and $(A-\lambda I)^\theta$ commute and hence $A-\lambda I$ is s-normal. Now $AX = \lambda X \Rightarrow (A-\lambda I)X = 0$. Since $A-\lambda I$ is s-normal, by theorem 3.4, $(A-\lambda I)^\theta X = 0$ which implies $(A^\theta - \overline{\lambda}I)X = 0$ and hence $A^\theta X = \overline{\lambda}X$ which leads to the result.

Theorem 3.6: If $A \in \mathcal{E}_{n \times n}$ is s-unitary and if $\lambda$ is an eigenvalue of $A$, then $|\lambda| = 1$.

Proof: Since $A \in \mathcal{E}_{n \times n}$ is s-unitary, $A$ is s-normal. Since $\lambda$ is an eigenvalue of $A$, there exists an eigenvector $V \neq 0$ such that $AV = \lambda V$ which implies $A^\theta V = \overline{\lambda}V$ as $A$ is s-normal. Now $V = IV = A^\theta AV$ which leads to $V(1 - \lambda\overline{\lambda}) = 0$. Since $V \neq 0$, $1 - \lambda\overline{\lambda} = 0$ which implies that $|\lambda| = 1$.

Theorem 3.7: Let $A \in \mathcal{E}_{n \times n}$. Assume that $A=VP$, where $V$ is s-unitary and $P$ is non singular and s-hermitian such that if $P^2$ commutes with $V$, then $P$ also commutes with $V$. Then the following conditions are equivalent.
Some equivalent conditions on s-normal matrices

(i) A is normal.
(ii) VP=PV
(iii) AV=VA
(iv) AP=PA

Proof:

Let A=VP. Since V is s-unitary $VV^0 = V^0V = I$ and since P is s-hermitian, $P^0 = P$.

(i) $\iff$ (ii): If A is s-normal, then $AA^0 = A^0A$. Since A=VP,

$$(VP)(VP)^0 = (VP)^0(VP)$$

which implies that $VP^2V^0 = P$. Post multiply by V,

we have $VP^2 = P^2V$ and hence $VP = PV$ by our assumption.

Conversely, if $VP = PV$, then $P^0V^0 = V^0P^0$.

Now $AA^0 = VPP^0V^0 = VPV^0P^0 = VP^0V^0P$ as $P^0 = P$. Therefore

$AA^0 = VV^0P^0P = V^0VPP = V^0PV = (PV)^0(VP) = (VP)^0(VP) = A^0A$

and hence A is s-normal.

(i) $\iff$ (iii): If A is s-normal, then by (ii), $VP = PV$.

Now $AV = (VP)V = V(VP) = VA$. Conversely, if $AV = VA$, then $(VP)V = V(VP)$,

pre multiply by $V^0$, $V^0V(VP) = V^0V(VP)$ which implies $PV = VP$ and hence

A is s-normal.

(i) $\iff$ (iv): If A is s-normal, then $AP = (VP)P = PVP = PA$.

Conversely, if $AP = PA$, then $(VP)P = P(VP)$. Post multiply by $P^{-1}$, we have $VP = PV$ and so

A is s-normal.

Theorem 3.8:

Let $A \in \mathcal{C}_{n \times n}$. Assume that A=VP, where V is s-unitary and P is non singular and s-hermitian such that if $P^2$ commutes with V, then P also commutes with V. Then the following conditions are equivalent.

(i) A is s-normal.
(ii) Any eigenvector of V is an eigenvector of P (as long as V has distinct eigen values)
(iii) Any eigenvector of P is an eigenvector of V (as long as P has distinct eigen values)
(iv) Any eigenvector of V is an eigenvector of A (as long as V has distinct eigen values)
(v) Any eigenvector of A is an eigenvector of V (as long as A has distinct eigen values)
(vi) Any eigenvector of \( P \) is an eigenvector of \( A \) (as long as \( P \) has distinct eigenvalues)

(vii) Any eigenvector of \( A \) is an eigenvector of \( P \) (as long as \( A \) has distinct eigenvalues)

**Proof:**

(i) \( \iff \) (ii):

Let \( V \) have distinct eigenvalues. If we prove \( VP = PV \iff \) any eigenvector of \( V \) is an eigenvector of \( P \), then (i) \( \iff \) (ii) follows by theorem 3.7. Assume that any eigenvector of \( V \) is an eigenvector of \( P \). If \( X \) is an eigenvector of \( V \), then \( X \) is also an eigenvector of \( P \). \( \therefore \) There exist eigenvalues \( \lambda \) and \( \mu \) such that \( VX = \lambda X \) and \( PX = \mu X \). Now \( VX = \lambda X \) implies \( PVX = P\lambda X = \lambda \mu X \). Similarly \( PX = \mu X \) implies \( VPX = \mu \mu X \). Therefore \( PVX = VPX \rightarrow (PV - VP)X = 0 \) which implies \( PV = VP \) as \( X \neq 0 \).

Conversely, assume that \( VP = PV \). If \( X \) is an eigenvector of \( V \), then there exists an eigenvalue \( \lambda \) such that \( VX = \lambda X \). Let \( \mu \) be an eigenvalue of \( V \) such that \( VX = \mu X \). \( \therefore \lambda \neq \mu \). Now \( VP = PV \) implies \( (PV - VP)X = 0 \) which shows that \( VPX = \lambda PX \). Similarly \( VX = \mu X \) implies \( VPX = \mu PX \). \( \therefore \lambda PX = \mu PX \rightarrow (\lambda - \mu)PX = 0 \Rightarrow PX = 0 \) as \( \lambda - \mu \neq 0 \). \( \therefore PX = 0X \) and hence \( X \) is an eigenvector of \( P \) corresponding to the eigenvalue 0. In general, if \( \mu \) is any eigenvalue of \( V \), then we can prove that \( X \) is also an eigenvector of \( P \). Therefore any eigenvector of \( V \) is also an eigenvector of \( P \).

Similar proof holds for other equivalent conditions.

**References**


**Received:** March, 2009