On an Equivalence of Simplicial Polytopes

Ashish Kumar Upadhyay

Department of Mathematics
Indian Institute of Technology Patna
Patliputra Colony, Patna - 800013 India
upadhyay@iitp.ac.in

Abstract

We show that a simplicial isomorphism of boundary polyhedra of simplicial polytopes extends to an equivalence of polytopes. Using this result we present another proof of the well known result that there are exactly five platonic solids.

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1 Introduction and Definitions

This article is an attempt to present a yet another simple proof of the well known classical theorem about classification of Platonic solids. Five geometrical objects, namely, the Tetrahedra, the Cube, the Octahedra, the Dodecahedra and the Icosahedra are called Platonic solids [2]. Extending continuous functions of spheres to the balls which they bound is a common exercise in a Topology graduate curriculum. But, whether such a result exists in the case of Combinatorial Topological category is not known to the author. We present such a result for the case of equivalence of simplicial polytopes here and use it to establish the result about classification. The existence is addressed by standard technique of constructing combinatorial 2-manifolds. We begin with some definitions.

An abstract n-simplex $\sigma$ is a finite set $\{v_1, v_2, \ldots, v_{n+1}\}$. An abstract n-simplex is also referred to as a n simplex. An abstract simplicial complex $K$ is a finite collection of abstract n-simplices for some positive integers $n$, together with all of their subsets. The greatest integer $d$ such that $K$ has a $d$ simplex is called the dimension of $K$. In this article we will be mainly dealing with
simplicial complexes of dimension 2. In such a case the elements of sizes 0, 1 and 2 are called vertices, edges and faces, respectively.

Let $\sigma$ be an abstract $n$-simplex, i.e. $\sigma = \{v_1, v_2, \ldots, v_{n+1}\}$. Let $\{x_1, x_2, \ldots, x_{n+1}\}$ be a set of geometrically independent (i.e. the set $\{x_i - x_1: 2 \leq i \leq n + 1\}$ is linearly independent) points in some Euclidean space $R^N$. Then, the convex hull $|\sigma|$ of $\{x_1, x_2, \ldots, x_{n+1}\}$ is known as geometric realization of $\sigma$ or geometric simplex corresponding to $\sigma$. Henceforth in this article we will omit the term abstract from abstract $n$-simplex and abstract simplicial complexes.

Let $K$ be a simplicial complex with the set of 0 simplices as $V = \{a_1, a_2, \ldots, a_m\}$. We can choose a positive integer $N$ sufficiently large and a set of points $U = \{b_1, b_2, \ldots, b_m\}$ in $R^N$ which is in one-to-one correspondence with $V$ such that the convex hull of $\{b_{i_1}, b_{i_2}, \ldots, b_{i_{r+1}}\} \subseteq U$ is a $r$-simplex in $R^N$ if $\{a_{i_1}, a_{i_2}, \ldots, a_{i_{r+1}}\} \subseteq V$ is a $r$-simplex in $K$. Then, the union of all the geometric simplices in $R^N$ corresponding to simplices in $K$ is called the geometric carrier of $K$ and is denoted by $|K|$. A simplicial map between two simplicial complexes $K_1$ and $K_2$ is a map $T$ of the vertex set $V(K_1)$ of $K_1$ into vertex set $V(K_2)$ of $K_2$ such that $\sigma \in K_1$ implies $T(\sigma) \in K_2$. The map $T$ is an isomorphism if $T: V(K_1) \rightarrow V(K_2)$ is a bijective map such that $T(\sigma)$ is a simplex if and only if $\sigma$ is a simplex. One may refer to [5] and [6] for further details about simplicial complexes.

A graph without loops and double edges is an example of 1-dimensional simplicial complex. The number of edges incident with a vertex $v$ in a simplicial complex is called the degree of $v$. A connected finite graph is called a cycle if the degree of each vertex is 2. An $n$-cycle is a cycle on $n$ vertices. It is denoted by $C_n(v_1, v_2, \ldots, v_n)$ if the edges are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ and $v_nv_1$.

If $v$ is a vertex of a simplicial complex $K$ then the link of $v$ in $K$ is the simplicial complex $\text{lk}(v) = \{\sigma \in K : v \notin \sigma, \{v\} \cup \sigma \in K\}$. A finite 2-dimensional simplicial complex $K$ is called a combinatorial 2-manifold if $|K|$ is a topological 2-manifold. It is easy to see that a simplicial complex $K$ is a combinatorial 2-manifold if and only if $\text{lk}(v)$ is a cycle for each vertex $v$ of $K$.

For an $n$-vertex combinatorial 2-manifold $M$ let $v, e$ and $f$ denote the numbers of vertices, edges and faces respectively. Then, the integer $\chi(M) = v - e + f$ is called the Euler characteristics of $M$. If the degree of each vertex in $M$ is same then it is called degree-regular. In following section we present some examples of degree-regular combinatorial 2-manifolds.

## 2 Examples

In this section we give pictorial examples of some orientable equivelar combinatorial 2-manifolds of Euler characteristics 2:
3 Results

Let $P$ be a convex subset of $\mathcal{R}^3$. A point $x \in P$ is called a vertex point (see [4], pp. 17) of $P$ if $y, z \in P, 0 < \lambda < 1$ and $x = \lambda y + (1 - \lambda)z$ imply $x = y = z$. The set of all vertex points of $P$ is called the vertex set of $P$ and is denoted as $vert P$. A compact convex set $P \subset \mathcal{R}^3$ is called a polytope if $vert P$ is a finite set. Thus, the platonic solids are examples of polytopes.

We say that a hyperplane $H = \{x \in \mathcal{R}^3 : \langle x, u \rangle = a, ||u|| = 1 \text{ and } a \neq 0\}$, supports $P$ if $H$ does not intersect $P$ but the distance $\delta(H, P)$ between $P$ and $H$ is 0, where $\delta(H, P) = \inf \{||b - a|| : a \in H, b \in P\}$. A set $F \subseteq P$ is called a face of $P$ if either $F = \emptyset$ or $F = P$ or there exists a supporting hyperplane $H$ of $P$ such that $F = P \cap H$ (see [4], pp. 17). The faces other than $\emptyset$ and $K$ are called proper faces of $P$. If all the proper faces of $P$ are simplices then $P$ is called a simplicial polytope.

An equivalence of two polytopes $P_1$ and $P_2$ (see [4], pp. 38) is a one - to - one map $T$ between the sets $\{F\}$ and $\{F'\}$ of faces of $P_1$ and $P_2$ respectively, which preserves inclusions. i.e. if $F_1 \subset F_2$ then $T(F_1) \subset T(F_2)$.

In Theorem 1 we show that a simplicial isomorphism of boundary polyhedra of a simplicial polytope extends to an equivalence of polytopes.

**Theorem 1.** Let $P_1$ and $P_2$ be two simplicial polytopes. Let $T : Bd(P_1) \longrightarrow Bd(P_2)$ be an isomorphism of the boundary polyhedra $Bd(P_1)$ and $Bd(P_2)$ of $P_1$ and $P_2$ respectively. Then $T$ extends to an equivalence $\tilde{T} : P_1 \longrightarrow P_2$.

**Proof.** By definition of $T$, $T(F)$ is a face of $P_2$ if and only if $F$ is a face of $P_1$, for all proper faces of $P_1$ and all the proper faces of $P_2$ are of this type. Now, extend $T$ by defining $\tilde{T}$ as $\tilde{T}(F) = T(F)$ for all proper faces $F$ of $P_1$ and
Lemma 1. Let \( M \) be a combinatorial 2–manifold on 12 vertices. If the degree of each vertex in \( M \) is 5, then \( M \) is isomorphic to \( I \) given in section 2 above.

Proof. Let the vertex set \( V \) of \( M \) be \( \{0, 1, \ldots, 11\} \). Assume without loss of generality that \( \text{lk}(0) = C_5(1, 2, 3, 4, 5) \). So, \( \text{lk}(1) \) has the form \( C_5(5, 0, 2, x, y) \), for some \( x, y \in V \). Clearly, \( x \notin \{0, 1, 2, 3, 5\} \).

If \( x = 4 \) then \( y \notin \{0, 1, 2, 4, 5\} \). If \( y \neq 3 \), then \( \text{lk}(4) \) contains 6 vertices, which is not possible. Hence \( y = 3 \). Then \( \text{lk}(1) = C_5(5, 0, 2, 4, 3) \) and so \( \text{lk}(4) = C_5(5, 0, 3, 1, 2) \). This implies \( \text{lk}(3) = C_5(2, 0, 4, 1, 5) \), \( \text{lk}(2) = C_5(3, 0, 1, 4, 5) \) and \( \text{lk}(5) = C_5(1, 0, 4, 2, 3) \). Then \( M \) is disconnected. Thus \( x \neq 4 \). So, we may assume that \( x = 6 \).

Now, \( y \notin \{0, 1, 2, 4, 5, 6\} \). If \( y = 3 \), then \( \text{lk}(3) \) contains 6 vertices, a contradiction. So, we may assume \( y = 7 \). Then \( \text{lk}(1) = C_5(2, 0, 5, 7, 6) \) and \( \text{lk}(2) \) has the form \( C_5(3, 0, 1, 6, z) \) for some \( z \in V \). It is easy to see that \( z \in \{5, 7, 8, 9, 10, 11\} \). If \( z = 5 \) then \( \text{lk}(5) \) has > 5 vertices. If \( z = 7 \) then \( C_5(7, 2, 1) \subseteq \text{lk}(6) \), a contradiction. So, we may assume that \( z = 8 \).

Thus \( \text{lk}(2) = C_5(3, 0, 1, 6, 8) \) and \( \text{lk}(3) \) has the form \( C_5(4, 0, 2, 8, w) \), for some \( w \in V \). It is easy to see that \( w \in \{6, 7, 9, 10, 11\} \). If \( w = 6 \) then \( C_5(2, 3, 6) \subseteq \text{lk}(8) \), a contradiction. If \( w = 7 \) then \( \text{lk}(7) \) has > 5 vertices. So, we may assume \( w = 9 \).

Thus \( \text{lk}(3) = C_5(4, 0, 2, 8, 9) \) and \( \text{lk}(4) \) has the form \( C_5(5, 0, 3, 9, u) \). A similar argument as in the previous case, shows that \( u = 10 \) or 11, say \( u = 10 \). So, \( \text{lk}(4) = C_5(5, 0, 3, 9, 10) \) and hence \( \text{lk}(5) = C_5(7, 1, 0, 4, 10) \). Now, \( \text{lk}(6) \) has the form \( C_5(7, 1, 2, 8, v) \) for some \( v \in V \). Since the vertices 0, 1, 2, 3, 4, 5 \notin \text{lk}(11), \ldots
it follows that $6 \in \text{lk}(11)$, i.e., $v = 11$. Hence $\text{lk}(6) = C_5(7, 1, 2, 8, 11)$. This implies $\text{lk}(7) = C_5(11, 6, 1, 5, 10)$, $\text{lk}(10) = C_5(9, 4, 5, 7, 11)$, $\text{lk}(8) = C_5(6, 2, 3, 9, 11)$, $\text{lk}(9) = C_5(11, 8, 3, 4, 10)$ and $\text{lk}(11) = C_5(7, 6, 8, 9, 10)$.

This is isomorphic to $\mathcal{I}$ by the map $T: M \rightarrow \mathcal{I}$ given by $T(0) = 0$, $T(1) = 7$, $T(2) = 8$, $T(3) = 9$, $T(4) = 10$, $T(5) = 11$, $T(6) = 4$, $T(7) = 3$, $T(8) = 5$, $T(9) = 1$, $T(10) = 2$ and $T(11) = 6$.

Of all the five polyhedra (see [1] for definitions and details of polyhedra) corresponding to boundary of platonic solids, three are simplicial, namely, the Tetrahedron, the Octahedron and the Icosahedron. It is well known that the Cube and Dodecahedron are dual polyhedron of Octahedron and Icosahedron respectively. Since, the Tetrahedron, the Octahedron and the Icosahedron are the only orientable simplicial degree-regular polyhedron (i.e. orientable combinatorial 2-manifolds) of Euler characteristic 2, it follows that up to isomorphism:

**Theorem 2.** There are exactly five orientable degree-regular polyhedra of Euler characteristic 2. These are, namely, the Tetrahedron, the Octahedron, the Cube, the Dodecahedron and the Icosahedron.

It follows from Theorem 1 and Theorem 2 that there are exactly five platonic solids.

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### References


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