Fuzzy Ideal Extensions of \( \Gamma \)-Semigroups via its Operator Semigroups

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Abstract
The main purpose of this paper is to explore new relationships between a \( \Gamma \)-semigroup and its operator semigroups in terms of fuzzy ideals so as to apply them, together with the already obtained such relationships, in the study of fuzzy ideal extension of \( \Gamma \)-semigroups.

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1 Introduction
The notion of fuzzy set was introduced by L.A.Zadeh[11], and since then this concept has been applied to various algebraic structures. In [3], Kuroki characterized several classes of semigroups in terms of fuzzy left, right and two-sided
ideals and bi-ideals. M. K. Sen and N. K. Saha[9] introduced the concept of Γ-semigroup as a generalization of semigroup and ternary semigroup. A natural example of a Γ-semigroup is the set of all mappings from any non-empty set \( A \) to a non-empty set \( B \) where \( \Gamma \) is the set of all mappings from \( B \) to \( A \) and ternary composition \( aab \) or \( a\alpha b \) \((a, b \in S, \alpha, \beta \in \Gamma)\) is the usual mapping composition. Uckun Mustafa, Ali Mehmet, Jun Young Bae[4] investigated some properties of fuzzy ideals in Γ-semigroups. T. K. Dutta and N. C. Adhikari[2] studied different properties of Γ-semigroup via its operator semigroups. In order to make the operator semigroups work in the context of fuzzy subsets of a Γ-semigroup as it did for conventional subsets we have investigated in [7], various relationships between the fuzzy ideals of a Γ-semigroup and that of its operator semigroups including an inclusion preserving bijection(Theorem 5.14.and 5.17[7]) between the set of all fuzzy ideals of a Γ-semigroup and that of its operator semigroups. Motivated by the work of Xie [10] on semigroups and the fact that Γ-semigroups encompass semigroups we extended the notion of fuzzy ideal extension to Γ-semigroups in [8]. The purpose of this paper is to bring operator semigroups into action to re-establish many results of [8]. To do this we first obtain some new relationships between a Γ-semigroup and its operator semigroups. These results together with those already obtained in [8] were used to work out as stated above. Among other results we have obtained characterization of a prime ideal of a Γ-semigroup in terms of fuzzy ideal extension.

2 Preliminaries

We recall the following definitions and results which will be used in the sequel.

**Definition 2.1** [2] Let \( S \) and \( \Gamma \) be two non-empty sets. \( S \) is called a Γ-semigroup if there exist mappings from \( S \times \Gamma \times S \) to \( S \), written as \((a, \alpha, b) \rightarrow aab\), and from \( \Gamma \times S \times \Gamma \) to \( \Gamma \), written as \((\alpha, a, \beta) \rightarrow a\alpha b\) satisfying the following associative laws \((aab)\beta c = a(ab\beta)\gamma = \alpha\beta a\alpha = \alpha a(\beta b)\gamma = \alpha \beta a = \alpha a + \beta (a, b \in S, \alpha, \beta \in \Gamma)\) for all \( a, b, c \in S \) and for all \( \alpha, \beta, \gamma \in \Gamma \).

**Example (a):** Let \( S \) be the set of all non-positive integers and \( \Gamma \) be the set of all non-positive even integers. Then \( S \) is a Γ-semigroup where \( aab \) and \( a\alpha b (a, b \in S, \alpha, \beta \in \Gamma)\) denote usual multiplication of integers.

**Example (b):** Let \( S = \{5n + 4 : n \text{ is a positive integer}\} \) and \( \Gamma = \{5n + 1 : n \text{ is a positive integer}\} \). Then \( S \) is a Γ-semigroup where \( aab = a + a + b \) and \( a\alpha b = a + a + \beta (a, b \in S, \text{ and } \alpha, \beta \in \Gamma) \), + is the usual addition of integers.

**Definition 2.2** [11] A fuzzy subset of a non-empty set \( X \) is a function \( \mu : X \rightarrow [0, 1] \).
Proposition 2.10

Definition 2.3 [10] The set of all fuzzy subsets of a set \( X \) with the relation \( f \subseteq g \Longleftrightarrow f(x) \leq g(x) \ \forall x \in X \) is a complete lattice where, for a non-empty family \( \{ \mu_i : i \in I \} \) of fuzzy subsets of \( X \), the \( \inf \{ \mu_i : i \in I \} \) and the \( \sup \{ \mu_i : i \in I \} \) are the fuzzy subsets of \( X \) defined by:

\[
\begin{align*}
\inf \{ \mu_i : i \in I \} : X & \rightarrow [0, 1], \ x \mapsto \inf \{ \mu_i(x) : i \in I \} \\
\sup \{ \mu_i : i \in I \} : X & \rightarrow [0, 1], \ x \mapsto \sup \{ \mu_i(x) : i \in I \}
\end{align*}
\]

Definition 2.4 [4] A non-empty fuzzy subset \( \mu \) of a \( \Gamma \)-semigroup \( S \) is called a fuzzy left ideal(right ideal) of \( S \) if \( \mu(x\gamma y) \geq \mu(y) \) (resp. \( \mu(x\gamma y) \geq \mu(x) \)) \( \forall x, y \in S, \ \forall \gamma \in \Gamma \).

Definition 2.5 [4] A non-empty fuzzy subset \( \mu \) of a \( \Gamma \)-semigroup \( S \) is called a fuzzy ideal of \( S \) if it is both fuzzy left ideal and fuzzy right ideal of \( S \).

Definition 2.6 [5] A fuzzy ideal \( \mu \) of a \( \Gamma \)-semigroup \( S \) is called a fuzzy prime ideal if \( \inf_{\gamma \in \Gamma} \mu(x\gamma y) = \max \{ \mu(x), \mu(y) \} \ \forall x, y \in S \).

Example (c): Let \( S \) be the set of all \( 1 \times 2 \) matrices over \( GF_2 \) (the finite field with two elements) and \( \Gamma \) be the set of all \( 2 \times 1 \) matrices over \( GF_2 \). Then \( S \) is a \( \Gamma \)-semigroup where \( a \circ b \) and \( \alpha \circ \beta(a, b \in S \) and \( \alpha, \beta \in \Gamma) \) denotes the usual matrix product. Let \( \mu : S \rightarrow [0, 1] \) be defined by

\[
\mu(x) = \begin{cases} 
0.3 & \text{if } a=\gamma_{0,0} \\
0.2 & \text{otherwise}
\end{cases}
\]

Then \( \mu \) is a fuzzy prime ideal of \( S \).

Definition 2.7 [6] A fuzzy ideal \( \mu \) of a \( \Gamma \)-semigroup \( S \) is called fuzzy semiprime ideal if \( \inf_{\gamma \in \Gamma} \mu(x\gamma x) \geq \mu(x) \ \forall x \in S \).

Proposition 2.8 [7] Let \( S \) be a \( \Gamma \)-semigroup and \( \phi \neq I \subseteq S \). Then \( I \) is an ideal of \( S \) iff \( \mu_I \) is a fuzzy ideal of \( S \), where \( \mu_I \) is the characteristic function of \( I \).

Definition 2.9 [2] Let \( S \) be a \( \Gamma \)-semigroup. Then an ideal \( I \) of \( S \) is said to be (i) prime if for ideals \( A, B \) of \( S \), \( A \Gamma B \subseteq I \) implies that \( A \subseteq I \) or \( B \subseteq I \). (ii) semiprime if for an ideal \( A \) of \( S \), \( A \Gamma A \subseteq I \) implies that \( A \subseteq I \).

Proposition 2.10 [5, 6] Let \( S \) be a \( \Gamma \)-semigroup and \( \phi \neq I \subseteq S \). Then \( I \) is a prime ideal(semiprime ideal) of \( S \) iff \( \mu_I \) is a fuzzy prime ideal(resp. fuzzy semiprime ideal) of \( S \), where \( \mu_I \) is the characteristic function of \( I \).

Theorem 2.11 [5, 6] Let \( I \) be an ideal of a \( \Gamma \)-semigroup \( S \). Then the following are equivalent:

(i) \( I \) is prime(semiprime).

(ii) for \( x, y \in S, x\Gamma y \subseteq I \Rightarrow x \in I \) or \( y \in I \) (resp. \( x\Gamma x \subseteq I \Rightarrow x \in I \)).

(iii) for \( x, y \in S, x\Gamma S \Gamma y \subseteq I \Rightarrow x \in I \) or \( y \in I \) (resp. \( x\Gamma S \Gamma x \subseteq I \Rightarrow x \in I \)).
Let $S$ be a $\Gamma$-semigroup, $\mu$ be a fuzzy subset of $S$ and $x \in S$, then the fuzzy subset $<x, \mu> : S \to [0, 1]$ defined by $<x, \mu>(y) = \inf_{\gamma \in \Gamma} \mu(x\gamma y)$ is called the extension of $\mu$ by $x$.

3 Corresponding Ideal Extensions

Let $S$ be a $\Gamma$-semigroup and $\rho$ a relation on $S \times \Gamma$ defined as : $(x, \alpha)\rho(y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Then $\rho$ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing $(x, \alpha)$. Let $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. Then $L$ is a semigroup with respect to the multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. This semigroup $L$ is called the left operator semigroup $L$ of the $\Gamma$-semigroup $S$.

Dually the right operator semigroup $R$ of $\Gamma$-semigroup $S$ is defined, where the multiplication is defined by $[\alpha, a][\beta, b] = [\alpha \beta, b]$. In [7], for fuzzy subsets of a $\Gamma$-semigroup $S$ and of its left, right operator semigroups $L, R$ respectively, we have defined four functions namely, $(\cdot)^{\ast}, (\cdot)^{+}, (\cdot)^{\ast'}, (\cdot)^{+'}$, which are defined as follows:

For a fuzzy subset $\mu$ of $R$ (or of $L$) a fuzzy subset $\mu^{\ast}$ (respectively $\sigma^{+}$) of $S$ by $\mu^{\ast}(a) = \inf_{\gamma \in \Gamma} \mu([\gamma, a])$ (respectively $\sigma^{+}(a) = \inf_{\gamma \in \Gamma} \sigma([a, \gamma])$) where $a \in S$ and for a fuzzy subset $\eta$ of $S$ we define a fuzzy subset $\eta^{\ast}$ (respectively $\eta^{+}$) of $L$ (respectively $R$) by $\eta^{\ast}([a, \alpha]) = \inf_{s \in S} \eta(a\alpha s)$ (respectively $\eta^{+}([\alpha, a]) = \inf_{s \in S} \eta(s\alpha a)$).

It has subsequently been proved that $(\cdot)^{\ast}, (\cdot)^{+}, (\cdot)^{\ast'}, (\cdot)^{+'}$ preserve fuzzy ideals. In fact we have shown that $(\cdot)^{\ast'}$ and $(\cdot)^{+'}$ are inclusion preserving bijections (cf. Theorem 5.14,5.13[7] with $(\cdot)^{\ast}$ and $(\cdot)^{+}$ are respectively their inverses. In [10] Xie defined for a semigroup $S$ and a fuzzy subset $\mu$ of $S$, a fuzzy subset $<x, \mu>$, called the extension of $\mu$ by $x$, by $<x, \mu>(y) := \mu(xy)$ $\forall y \in S$. It is also proved that if $\mu$ is a fuzzy ideal of $S$ then so is $<x, \mu>$.

Now we can deduce the following propositions easily:

**Proposition 3.1** Let $S$ be a commutative $\Gamma$-semigroup and $L(R)$ the left (respectively the right) operator semigroups of $S$. Let $\mu$ be a fuzzy (left, right, two sided) ideal of $S$ then $<x, \mu^{+}>$ (respectively $<x, \mu^{\ast}>$) is a fuzzy (left, right, two sided) ideal of $L$ (respectively $R$) for all $x \in L(R)$.

**Proposition 3.2** (With the same notation as in the above proposition) If $\sigma$ is a fuzzy (left, right, two sided) ideal of $L(R)$ then $<x, \sigma^{+}>$ (respectively $<x, \sigma^{\ast}>$) is a fuzzy (left, right, two sided) ideal of $S$ for all $x \in S$.
Lemma 3.3 Let $\mu$ be a non-empty fuzzy subset of a commutative $\Gamma$-semigroup $S$. Then for all $x \in S$

(i) $< x,\mu >^* \subseteq < [\alpha, x],\mu >^* \forall \alpha \in \Gamma$.

(ii) $< x,\mu >^* = \inf_{\alpha \in \Gamma} < [\alpha, x],\mu >^*$.

Proof. (i) Let $[\beta, y] \in R$. Then

\[
< x,\mu >^* ([\beta, y]) = \inf_{s \in S} < x,\mu > (s\beta y)
\]

\[
= \inf_{s \in S} \inf_{\gamma \in \Gamma} \mu(x\gamma s\beta y) = \inf_{\gamma \in \Gamma} \inf_{s \in S} \mu(x\gamma s\beta y)
\]

Again

\[
< [\alpha, x],\mu >^* ([\beta, y]) = \mu^*([\alpha, x][\beta, y])
\]

\[
= \mu^*([\alpha, x\beta y]) = \inf_{s \in S} (s\alpha x\beta y)
\]

\[
= \inf_{\gamma \in \Gamma} \inf_{s \in S} \mu(x\alpha s\beta y)\text{(using commutativity of } S)\]

Since, $\inf_{\gamma \in \Gamma} \inf_{s \in S} \mu(x\gamma s\beta y) \leq \inf_{s \in S} \mu(x\alpha s\beta y)$, we have

\[
< x,\mu >^* ([\beta, y]) \leq < [\alpha, x],\mu >^* ([\beta, y])
\]

whence $< x,\mu >^* \subseteq < [\alpha, x],\mu >^*$.

(ii) Let $[\beta, y] \in R$. Then

\[
\inf_{\alpha \in \Gamma} < [\alpha, x],\mu >^* ([\beta, y]) = \inf_{\alpha \in \Gamma} \mu^*([\alpha, x][\beta, y])
\]

\[
= \inf_{\alpha \in \Gamma} \mu^*([\alpha, x\beta y]) = \inf_{\alpha \in \Gamma} \inf_{s \in S} \mu(s\alpha x\beta y)
\]

\[
= \inf_{s \in S} < x,\mu > (s\beta y) = < x,\mu >^* ([\beta, y])
\]

Thus $< x,\mu >^* = \inf_{\alpha \in \Gamma} < [\alpha, x],\mu >^*$.

Lemma 3.4 Let $\sigma$ be a non-empty fuzzy subset of the right operator semigroup $R$ of a $\Gamma$-semigroup $S$ then for all $x \in S$, $< [\beta, x],\sigma >^* \subseteq < x,\sigma >^* \forall \beta \in \Gamma$.

Proof. Let $p \in S$. Then

\[
< [\beta, x],\sigma >^* (p) = \inf_{\gamma \in \Gamma} < [\beta, x],\sigma > ([\gamma, p])
\]

\[
= \inf_{\gamma \in \Gamma} \sigma([\beta, x][\gamma, p]) = \inf_{\gamma \in \Gamma} \sigma([\beta, x\gamma p])
\]
Again
\[
< x, \sigma^* > (p) = \inf_{\gamma \in \Gamma} \sigma^*(x \gamma p) = \inf_{\gamma \in \Gamma; \beta \in \Gamma} \sigma((\beta, x \gamma p)) \\
= \inf_{\beta \in \Gamma; \gamma \in \Gamma} \sigma((\beta, x \gamma p))
\]

Thus we have \( < [\beta, y], \sigma^* > (p) \geq < x, \sigma^* > (p) \). Consequently, \( < [\beta, y], \sigma^* > (p) \geq < x, \sigma^* > (p) \).

For the sake of convenience, from [1] we recall here that there are four functions namely \((\cdot)^+, (\cdot)^+, (\cdot)^*, (\cdot)^*\) for a \(\Gamma\)-semigroup \(S\) and its left and right operator semigroups \(L, R\) respectively. They are defined as follows: For \( I \subseteq R, I^* = \{ s \in S, [\alpha, s] \in J \forall \alpha \in \Gamma \}; \) for \( P \subseteq S, P^* = \{ [\alpha, x] \in R : s \alpha x \in P \forall s \in S \}; \) for \( J \subseteq L, J^+ = \{ s \in S, [s, \alpha] \in J \forall \alpha \in \Gamma \}; \) for \( Q \subseteq S, Q^+ = \{ [x, \alpha] \in L : x \alpha s \in Q \forall s \in S \}. \)

**Lemma 3.5** Let \( I \) be an ideal, \( \mu \) a fuzzy ideal of a \(\Gamma\)-semigroup \(S\) and \( R \) the right operator semigroup of \(S\). Then \((\mu_I)^\ast = \mu_I^*\), where \(\mu_I\) is the characteristic function of \(I\).

**Proof.** Let \( [\beta, y] \in R \). Then \((\mu_I)^*([\beta, y]) = \inf_{s \in S} \mu(s \beta y)\). Suppose \( [\beta, y] \in I^*\).

Then \( s \beta y \in I \) for all \( s \in S \). Hence \( \mu_I(s \beta y) = 1 \) for all \( s \in S \). This gives \( \inf_{s \in S} \mu(s \beta y) = 1 \) whence \((\mu_I)^*([\beta, y]) = 1\). Also \( \mu_I^*([\beta, y]) = 1\). Hence \((\mu_I)^*([\beta, y]) = \mu_I^*([\beta, y])\). Suppose \( [\beta, y] \not\in I^*\). Then for some \( t \in S, t \beta y \not\in I\).

So \( \mu_I(t \beta y) = 0\). This gives \( \inf_{s \in S} \mu_I(s \beta y) = 0 \) i.e., \((\mu_I)^*([\beta, y]) = 0\). Again \( \mu_I^*([\beta, y]) = 0\). Thus in this case also \((\mu_I)^*([\beta, y]) = \mu_I^*([\beta, y])\). Hence \((\mu_I)^* = \mu_I^*\).

**Lemma 3.6** Let \( \{ A_\alpha \}_{\alpha \in \Lambda} \) be a family of ideals of a \(\Gamma\)-semigroup \(S\). Then \((\bigcap_{\alpha \in \Lambda} A_\alpha)^* = \bigcap_{\alpha \in \Lambda} A_\alpha^*\).

**Proof.** Let \( [\alpha, x] \in (\bigcap_{\alpha \in \Lambda} A_\alpha)^*\). Then \( s \alpha x \in \bigcap_{\alpha \in \Lambda} A_\alpha \forall s \in S\). Hence \( \forall s \in S, s \alpha x \in A_\alpha \forall \alpha \in \Lambda\). This gives \( [\alpha, x] \in \bigcap_{\alpha \in \Lambda} A_\alpha^*\). Hence \((\bigcap_{\alpha \in \Lambda} A_\alpha)^* \subseteq \bigcap_{\alpha \in \Lambda} A_\alpha^*\). Reversing the above argument we deduce that \( \bigcap_{\alpha \in \Lambda} A_\alpha^* \subseteq (\bigcap_{\alpha \in \Lambda} A_\alpha)^*\).

Hence \((\bigcap_{\alpha \in \Lambda} A_\alpha)^* = \bigcap_{\alpha \in \Lambda} A_\alpha^*\).

**Lemma 3.7** Let \( S \) be a \(\Gamma\)-semigroup, \( R \) its right operator semigroup and \( \mu := \inf \{ \mu_i : i \in I \} \) a non-empty family of fuzzy subsets of \(S\). Then \( \mu^* = \inf \{ \mu_i^* : i \in I \} \).
Proof. Let \([\alpha, x] \in R\). Then
\[
\mu^{s'}[\alpha, x] = (\inf\{\mu_i : i \in I\})^{s'}[\alpha, x]
\]
\[
= \inf_s (\inf_{i \in I}\{s\alpha x \})
\]
Now
\[
\inf_i \{\mu_i^{s'} : i \in I\}([\alpha, x]) = \inf_{i \in I}(\mu_i^{s'}[\alpha, x])
\]
\[
= \inf_{i \in I} \inf_s (s\alpha x)
\]
Hence the lemma. ■

Now by using the above lemmas together with results obtained in [5],[6] on correspondence between the fuzzy prime(semiprime) ideals of \(S\) and that of the operator semigroups and theorem 5.14,5.13[7] we deduce the following results on fuzzy ideal extension of \(\Gamma\)-semigroups

**Proposition 3.8** Let \(S\) be a commutative \(\Gamma\)-semigroup with unities and \(\mu\) a fuzzy ideal of \(S\). Then \(<x, \mu>\) is a fuzzy ideal of \(S\) for all \(x \in S\).

**Proof.** Let \(R\) be the right operator semigroup of \(S\). Since \(S\) is commutative, its right operator semigroup \(R\) is also commutative. Now \(\mu^{s'}\) is a fuzzy ideal of \(R\)(cf. Proposition 5.10[7]). Let \(x \in S\). Then for any \(\alpha \in \Gamma\), \(<[\alpha, x], \mu^{s'}>\) is a fuzzy ideal of \(R\)(cf. Proposition 3.2[10]) and hence \(\inf_{\alpha \in \Gamma} \{[\alpha, x], \mu^{s'}\}\) is a fuzzy ideal of \(R\). This together with Lemma 3.3 implies that \(<x, \mu>^{s'}\) is a fuzzy ideal of \(R\). Consequently, \((<x, \mu>^{s'})^*\) is a fuzzy ideal of \(S\) (cf. Proposition 5.9[7]), i.e., \(<x, \mu>\) is a fuzzy ideal of \(S\)(cf. Theorem 5.13,5.14[7]). ■

Now by applying Proposition 5.3[3], Corollary 3.9[8] and Lemma 3.3 we deduce the following proposition.

**Proposition 3.9** Let \(S\) be a commutative \(\Gamma\)-semigroup. If \(\mu\) is a fuzzy semiprime ideal of \(S\), then \(<x, \mu>\) is a fuzzy semiprime ideal of \(S\) for every \(x \in S\).

**Theorem 3.10** Let \(S\) be a commutative \(\Gamma\)-semigroup, \(\{\mu_i\}_{i \in I}\) be a non-empty family of fuzzy semiprime ideals of \(S\) and let \(\mu = \inf\{\mu_i : i \in I\}\). Then for any \(x \in S\), \(<x, \mu>\) is a fuzzy semiprime ideal of \(S\).

**Proof.** Let \(R\) be the right operator semigroup of \(S\). Since \(S\) is commutative, its right operator semigroup \(R\) is commutative. Now \(\{\mu_i^{s'}\}_{i \in I}\) is a non-empty family of fuzzy semiprime ideals of \(R[6]\). Hence \(\inf\{\mu_i^{s'}\}_{i \in I}\) is a fuzzy semiprime
ideal of $R$. Again by Lemma 3.7, $\inf \{\mu_i\}_{i \in I} = \mu^*$. Thus we see that $\mu^*$ is a fuzzy semiprime ideal of $R$. This means that for any $[\alpha, x] \in R$, $< [\alpha, x], \mu^* >$ and hence $\inf_{\alpha \in \Gamma} < [\alpha, x], \mu^* >$ is a fuzzy semiprime ideal of $R$. This together with the Lemma 3.3 implies that $< x, \mu >^*$ is a fuzzy semiprime ideal of $R$. Hence $(< x, \mu >^*)^{[6]}$ i.e., $< x, \mu >$ (cf. Theorem 5.14[7]) is a fuzzy semiprime ideal of $S$.

**Theorem 3.11** Let $S$ be a commutative $\Gamma$-semigroup, $\{S_i\}_{i \in I}$ a non-empty family of semiprime ideals of $S$ and $A := \cap_{i \in I}S_i \neq \phi$. Then $< x, \mu_A >$ is a fuzzy semiprime ideal of $S$ for all $x \in S$ where $\mu_A$ is the characteristic function of $A$.

**Proof.** Since $\forall i \in I, S_i$ is a prime ideal of $S$, $S_i^*$ is a prime ideal of the right operator semigroup $R$ of $S, \forall i \in I[6]$. Now $A := \cap_{i \in I}S_i. A^* = (\cap_{i \in I}S_i)^* = \cap_{i \in I}S_i^*$ (cf. Lemma 3.6) $\neq \phi$. So by Corollary 3.11[10], $< [\alpha, x], \mu_{A^*} >$ is a fuzzy semiprime ideal of $R, \forall \alpha \in \Gamma$. Hence $\inf_{\alpha \in \Gamma} < [\alpha, x], \mu_{A^*} >$ is a fuzzy semiprime ideal of $R$. This together with Lemma 3.5 implies that $\inf_{\alpha \in \Gamma} < [\alpha, x], (\mu_A)^* >$, i.e., $< x, \mu_A >^*$ (cf. Lemma 3.3(ii)) is a fuzzy semiprime ideal of $R$. Hence $(< x, \mu_A >^*)^*$ is a fuzzy semiprime ideal of $S[6]$. Consequently, by Theorem 5.14[7], $< x, \mu_A >$ is a fuzzy semiprime ideal of $S$.

**Theorem 3.12** Let $S$ be a $\Gamma$-semigroup and $I$ be an ideal of $S$. Then $I$ is prime if and only if for $x \in S$ with $x \notin I$, $< x, \mu_I > = \mu_I$, where $\mu_I$ is the characteristic function of $I$.

**Proof.** Let $I$ be a prime ideal of $S$ and $x \notin I$. Then by dual of Lemma 3.12[2], $I^*$ is a prime ideal of the right operator semigroup $R$ of $S[1]$. Also $[\alpha, x] \notin I^*$ for some $\alpha \in \Gamma$. Then $< [\alpha, x], \mu_{I^*} > = \mu_{I^*} >$ (cf. Corollary 3.6[10]).

Now $(\mu_I)^* = \mu_{I^*}$ (cf. Lemma 3.5). It follows that $< [\alpha, x], \mu_{I^*} > = (\mu_I)^*$. Hence $< [\alpha, x], \mu_{I^*} > = ((\mu_I)^*)^* = \mu_I$ (cf. Theorem 5.13[7]). Now by Lemma 3.4, $< x, (\mu_I)^* > \subseteq < [\alpha, x], \mu_{I^*} >^*$. Hence $< x, (\mu_I)^* > \subseteq \mu_I$. This together with Lemma 3.5 implies that $< x, (\mu_I)^* > \subseteq \mu_I$. Hence by applying Theorem 5.14[7] we deduce that $< x, \mu_I > \subseteq \mu_I$. Also $\mu_I \subseteq < x, \mu_I >$ (cf. Proposition 3.7[5]). Hence $< x, \mu_I > = \mu_I$.

Conversely, suppose $< z, \mu_I > = \mu_I$ for all $z$ in $S$ with $z \notin I$. Let $x \Gamma y \subseteq I$ where $x, y \in S$. Then $\mu_I(x\gamma y) = 1 \forall \gamma \in \Gamma$. Let $x \notin I$. Then by hypothesis $< x, \mu_I > = \mu_I$. This gives $< x, \mu_I > (y) = \mu_I(y)$, i.e., $\inf_{\gamma \in I} \mu_I(x\gamma y) = \mu_I(y)$. Hence $\mu_I(y) = 1$ whence $y \in I$. Consequently $I$ is a prime ideal of $S$ (cf. Theorem 3.12).
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