On Maximum Indexable Graphs

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Abstract

Let $G = (V, E)$ be a $(n, m)$ graph. $G$ is said to be maximum indexable if there exists a bijection $f : V \rightarrow \{0, 1, 2, \cdots , n - 1\}$ such that $f^{\max} : E \rightarrow N$ is injective, where $f^{\max}(uv) = f(u) + f(v) + \max\{f(u), f(v)\}$. In this paper we prove that all trees and all unicyclic graphs are maximum indexable. We also construct several classes of maximum indexable graphs. We derive an explicit formula for $\lambda(n)$, the maximum number of edges in a maximum indexable graph of order $n$.

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1 INTRODUCTION

Let $G = (V, E)$ be a $(n, m)$ graph. We assume that $G$ is a finite, undirected, connected graph without loops or multiple edges. Graph labelings, where the vertices are assigned certain values subject to some conditions, have often been motivated by practical problems. In the last five decades an enormous work has been done on this subject. The concept of indexable graph was introduced by Acharya and Hegde [1]. They have conjectured that all unicyclic graphs are indexable. In 1996, Arumugam and Germina [3] proved this conjecture and also proved that all trees are indexable using breadth first search (BFS) algorithm [2]. They have also constructed several classes of
strongly indexable *unicyclic* graphs. Motivated by these works, in this paper we introduce the concept of maximum indexable graphs and prove that all *unicyclic* graphs are maximum indexable and also prove that all trees are maximum indexable. We derive an explicit formula for $\lambda(n)$, the maximum number of edges in a maximum indexable graph of order $n$ and also construct several classes of maximum indexable graphs.

**Definition 1.1.** A labeling of $G$ is a bijection $f : V \rightarrow \{0, 1, 2, \ldots, n-1\}$. For any edge $e = uv$ of $G$, let $f_{\text{max}}(e) = f(u) + f(v) + \max\{f(u), f(v)\}$. $G$ is said to be maximum indexable if there exists a labelling $f$ of $G$ such that $f_{\text{max}} : E \rightarrow \mathbb{N}$ is injective.

**Example 1.2.** The Peterson graph is a maximum indexable graph.

![Peterson Graph](image)

**Figure 1**

### 2 MAIN RESULTS

In this section, we prove that all trees and all *unicyclic* graphs are maximum indexable.

**Theorem 2.1.** All *trees* are maximum indexable.

**Proof.** Let $v$ be any vertex of a tree $T$. Starting from $v$, we visit all the vertices of $T$ using BFS algorithm and label the vertices with the consecutive
numbers 0, 1, 2, · · · , (n−1) in the order in which they are visited. Let \( e_1 = v_1v_2 \)
and \( e_2 = v_1v_3 \) be two adjacent edges of \( T \). Then

1. \( f(v_1) < f(v_2) < f(v_3) \Rightarrow f_{\text{max}}(e_1) < f_{\text{max}}(e_2) \),
2. \( f(v_1) < f(v_3) < f(v_2) \Rightarrow f_{\text{max}}(e_1) > f_{\text{max}}(e_2) \),
3. \( f(v_2) < f(v_3) < f(v_1) \Rightarrow f_{\text{max}}(e_1) < f_{\text{max}}(e_2) \),
4. \( f(v_2) < f(v_1) < f(v_3) \Rightarrow f_{\text{max}}(e_1) < f_{\text{max}}(e_2) \),
5. \( f(v_3) < f(v_1) < f(v_2) \Rightarrow f_{\text{max}}(e_1) > f_{\text{max}}(e_2) \),
6. \( f(v_3) < f(v_2) < f(v_1) \Rightarrow f_{\text{max}}(e_1) > f_{\text{max}}(e_2) \).

Thus in all cases \( f_{\text{max}}(e_1) \neq f_{\text{max}}(e_2) \). Suppose \( e_1 = v_1v_2 \) and \( e_2 = v_3v_4 \) are
non-adjacent edges of \( T \). Let \( f(v_1) < f(v_2), f(v_3), f(v_4) \). Now since \( T \) is a
tree at least one of the vertices \( v_3, v_4 \) is not adjacent to \( v_1 \). Suppose \( v_4 \) is not
adjacent to \( v_1 \), then \( f(v_2) < f(v_4) \). Note that

\[
f_{\text{max}}(e_1) = f(v_1) + 2f(v_2)
\]

and

\[
f_{\text{max}}(e_2) = \begin{cases} f(v_3) + 2f(v_4) & \text{if } f(v_3) < f(v_4), \\ f(v_4) + 2f(v_3) & \text{if } f(v_4) < f(v_3). \end{cases}
\]

Hence it follows that \( f_{\text{max}}(e_1) < f_{\text{max}}(e_2) \). This completes the proof.

**Theorem 2.2.** Every cycle is maximum indexable.

**Proof.** We show that every cycle \( C_k = (v_1, v_2, \cdots, v_k, v_1) \) is maximum indexable. Starting from \( v_1 \), we visit all the vertices of \( C_k \) using BFS algorithm
and label the vertices with the consecutive integers 0, 1, · · · , \( k−1 \) in the order
in which they are visited. Clearly this labeling shows that \( C_k \) is maximum indexable.

**Theorem 2.3.** Every unicyclic graph is maximum indexable.

**Proof.** As the method of proof is similar to that used in [3], we omit the proof.
3 SOME FAMILIES OF MAXIMUM INDEXABLE GRAPHS

In this section we show that certain well known families of graphs like ladder, triangular ladder and $P_2 \times C_n$ are maximum indexable graphs.

Theorem 3.1. Every Ladder is maximum indexable graph.

Proof. We label the vertices of a ladder as shown in the following figure.

We can arrange the values of edges of ladder in an increasing sequence

\[ \{2, 4, 7, 8, 10, 13, \ldots, 6n - 8, 6n - 5, 6n - 4\}. \]

Thus ladder is a maximum indexable graph.
Definition 3.2. A path of length $n - 1$, denoted by $P_n$, is a sequence of distinct edges $v_1, v_2, v_3, \ldots, v_{n-1}v_n$ with $v_iv_{i+1} \in E(P_n)$. A closed path, with $v_1 = v_n$, is called a cycle or a circuit and denoted by $C_{n-1}$.

Theorem 3.3. $P_2 \times C_n$ is a maximum indexable graph.

Proof. We consider two cases.

Case (i): $n = 2m$. In this case we label the vertices of $P_2 \times C_{2m}$ as shown in the Figure 3.

![Figure 3](image_url)

Figure 3

Case (ii): If $n = 2m + 1$ We label the vertices of $P_2 \times C_{2m+1}$ as shown in Figure 4.
It is clear that the value of edges of $P_2 \times C_n$ are all distinct, and hence $P_2 \times C_n$ is maximum indexable graph.

**Definition 3.5.** The join $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

**Theorem 3.6.** The graph $G = K_2 + \overline{K_{n-2}}$ for $n \geq 3$ is a maximum indexable graph.

**Proof.** Let the vertices of $K_2$ be $v_1$ and $v_n$ and those of $\overline{K_{n-2}}$ be $v_2, v_3, \ldots, v_{n-1}$. Define the map $f : V(G) \rightarrow \{0, 1, \ldots, n-1\}$ by letting $f(v_i) = i - 1$, $1 \leq i \leq n$. We can arrange the values of edges of $K_2 + \overline{K_{n-2}}$ in an increasing sequence $\{2, 4, \ldots, (2n - 4), (2n - 2), (2n - 1), 2n, \ldots, 3n - 5, 3n - 4\}$. Thus $K_2 + \overline{K_{n-2}}$ is a maximum indexable graph.
4 MAXIMUM NUMBER OF EDGES IN A MAXIMUM INDEXABLE GRAPH

In this section we establish a formula for \( \lambda(n) \), the maximum number of edges in a maximum indexable graph of order \( n \).

**Theorem 4.1.** Let \( \lambda(n) \) denote the maximum number of edges in a maximum indexable graph of order \( n \). Then \( \lambda(2) = 1 \) and \( \lambda(n) = 3n - 6 \), \( n \geq 3 \).

**Proof.** It is trivial to check that \( \lambda(2) = 1 \). So we assume that \( n \geq 3 \). Let \( S_k = \{r + 2s \mid 0 \leq r < s \leq k\} \). Then it is clear that \( S_1 \subset S_2 \subset S_3 \subset \cdots \subset S_{n-1} \) and \( \lambda(n) = |S_{n-1}| \). Define \( A_1 = S_1 \) and \( A_k = S_k - S_{k-1}, k \geq 2 \). Then
\[ A_i \cap A_j = \emptyset \text{ for } i \neq j \text{ and } S_{n-1} = \bigcup_{k=1}^{n-1} A_k \text{ and hence} \]

\[ \lambda(n) = |S_{n-1}| = \sum_{k=1}^{n-1} |A_k| = 1 + 2 + \sum_{k=3}^{n-1} |A_k|. \]

Since \( r + 2k = (r + 2) + 2(k - 1) \in S_{k-1} \) for \( r = 0, 1, 2, \ldots, k - 4 \), we have \( r + 2s \in A_k \) if and only if \( k - 3 \leq r < s = k \). Thus \( |A_k| = 3, k \geq 3 \). Hence

\[ \lambda(n) = 3 + \sum_{k=3}^{n-1} |A_k| = 3 + \sum_{k=3}^{n-1} 3 = 3n - 6. \]

\section*{REFERENCES}


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