Uniform Convergence of Schwarz Method for Noncoercive Variational Inequalities

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Abstract

In this paper we study noncoercive variational inequalities studied by Courty-Dumont, using the Schwarz method. The main idea of this method consists in decomposing the domain in two subdomains. We demonstrate that the discretisation on every subdomain converges in uniform norm and we give a result of approximation for the method in uniform norm.

Mathematics Subject Classification: 05C38, 15A15; 05A15, 15A18

Keywords: Schwarz method, Variational inequalities, \( L^\infty \)-error estimates.

1 Introduction

We are interested in the following noncoercive variational inequality. Find \( u \in H^1_0 (\Omega) \) solution of

\[
\begin{align*}
& a (u, v - u) \geq (f, v - u) \\
& u \leq \Psi, v \leq \Psi
\end{align*}
\]

(1)

where \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^2 \) with boundary \( \partial \Omega \).

and the noncoercive bilinear form \( a (u, v) \).

or equivalently. Find \( u \in H^1_0 (\Omega) \) solution of

\[
\begin{align*}
& b (u, v - u) \geq (f + \lambda u, v - u) \\
& u \leq \Psi, v \leq \Psi
\end{align*}
\]

(2)

where

\[
b (u, v) = a (u, v) + \lambda (u, v)
\]

(3)
and \( \lambda > 0 \) large enough such that \( \forall v \in H^1_0(\Omega) \) we have

\[
b(v, v) \geq \mu \|v\|^2_{H^1(\Omega)}, \mu > 0
\]

(4)

In section 2, we give the continuous V.I problem, we study the existence and the uniqueness of the solution, then we introduce the continuous Schwarz method. In section 3, we consider the discrete problem and we establish a survey similar to the one of the continuous case. In section 4, we establish the error estimates in the \( L^\infty \) norm for the problem studied.

## 2 The Continuous Problem

### 2.1 Notations and Assumptions

Let’s consider the functions

\[
a_{i,j}(x), a_i(x), a_0(x) \in C^2(\overline{\Omega}), x \in \overline{\Omega}, 1 \leq i, j \leq n
\]

(5)

such that

\[
\sum_{1 \leq i,j \leq n} a_{ij}(x)\xi_i\xi_j \geq \alpha |\xi|^2; \xi \in \mathbb{R}^n, \alpha > 0
\]

(6)

\[
a_{ij}(x) = a_{ji}(x); a_0(x) \geq \beta > 0
\]

(7)

We define the bilinear form, \( \forall u, v \in H^1_0(\Omega) \)

\[
a(u, v) = \int_{\Omega} \left( \sum_{1 \leq i,j \leq n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{1 \leq i \leq n} a_i(x) \frac{\partial u}{\partial x_i} v + a_0(x)uv \right) dx
\]

(8)

Let \( f \) be

\[
f \in L^\infty(\Omega) \cap C^2(\overline{\Omega}); f \geq 0
\]

(9)

and on \( \Omega \)

\[
K(\Psi, g) = \{ v \in H^1(\Omega), v - g \in H^1_0(\Omega), 0 \leq v \leq \Psi \}
\]

(10)

with the obstacle \( \Psi \) and \( g \) is a regular function defined on \( \partial \Omega \).

\[
\Psi, g \in W^{2,p}(\Omega), p > 2; o \leq g \leq \Psi
\]

(11)
2.2 The Continuous Problem

Find $u \in K(\Psi, g)$ the solution of

$$b(u, v - u) \geq (f + \lambda u, v - u), \forall v \in K(\Psi, g)$$  \hspace{1cm} (12)

**Theorem 2.1** ([10]) Under the conditions (5) to (11), the problem (12) has an unique solution $u \in K(\Psi, g)$. Moreover we have

$$u \in W^{2,p}(\Omega), 2 \leq p < \infty$$  \hspace{1cm} (13)

Denote the solution $u$ of the variational inequality (12) by $\sigma(f, g)$.

We introduce the property of the monotonicity of the solution of (12).

**Proposition 2.2** Under the hypotheses (5) to (11) and previous notations, we have if $f \leq \tilde{f}$ and $g \leq \tilde{g}$ then $\sigma(f, g) \leq \sigma(\tilde{f}, \tilde{g})$

**Proof.** Let $f \leq \tilde{f}$ and $g \leq \tilde{g}$.

Let’s put $u = \sigma(f, g)$ and $\tilde{u} = \sigma(\tilde{f}, \tilde{g})$, we have

$$a(u, v) \leq (f, v), \forall u \in K(\Psi, g), v \geq 0$$

therefore

$$a(u, v) \leq (f, v) \leq (\tilde{f}, v), \forall u \in K(\Psi, g), v \geq 0$$

thus

$$a(u, v) \leq (\tilde{f}, v), \forall u \in K(\Psi, g), v \geq 0$$

then, $u$ is one subsolution for the solution $w = \sigma(\tilde{f}, g)$.

As $w$ is the biggest subsolution, we have

$$u = \sigma(f, g) \leq w = \sigma(\tilde{f}, g)$$

One knows that $\sigma(\tilde{f}, g) \leq \sigma(\tilde{f}, \tilde{g})$.

then

$$u = \sigma(f, g) \leq \tilde{u} = \sigma(\tilde{f}, \tilde{g})$$

hence

$$\sigma(f, g) \leq \sigma(\tilde{f}, \tilde{g})$$

**Remark 2.3** If $g = \tilde{g}$ we have $\sigma(f) \leq \sigma(\tilde{f})$.

We show an important proposition. Which give the Continuous dependence to the data $g$ and $f$ contrary to those in [1] that are $g$ and $\Psi$. 
Proposition 2.4 Under the hypotheses (5) to (11) and previous notations, we have
\[ \|u - \tilde{u}\|_{L^\infty(\Omega)} \leq C \left( \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \|f - \tilde{f}\|_{L^\infty(\Omega)} \right) \] (14)
where \( C \) is an independent constant of \( f \) and \( g, Ca_0(x) \geq 1 \).

Proof. Let’s put
\[ \Phi = C \left( \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \|f - \tilde{f}\|_{L^\infty(\Omega)} \right) \]
We have
\[ \tilde{f} \leq f + \left( \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \|f - \tilde{f}\|_{L^\infty(\Omega)} \right) \]
\[ \leq f + Ca_0 \left( \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \|f - \tilde{f}\|_{L^\infty(\Omega)} \right) \]
\[ \leq f + a_0 \Phi \]
and
\[ \tilde{g} \leq g + C \left( \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \|f - \tilde{f}\|_{L^\infty(\Omega)} \right) = g + \Phi \]
As
\[ \sigma (f + a_0 \Phi, g + \Phi) = \sigma (f, g) + \Phi \]
we have
\[ \sigma (\tilde{f}, \tilde{g}) \leq \sigma (f, g) + \Phi \]
then
\[ \sigma (\tilde{f}, \tilde{g}) - \sigma (f, g) \leq \Phi \]
Since \((\tilde{f}, \tilde{g})\) and \((f, g)\) are symmetrical, we have also
\[ \sigma (f, g) - \sigma (\tilde{f}, \tilde{g}) \leq \Phi \]
then
\[ \|u - \tilde{u}\|_{L^\infty(\Omega)} \leq C \left( \|g - \tilde{g}\|_{L^\infty(\partial\Omega)} + \|f - \tilde{f}\|_{L^\infty(\Omega)} \right) . \]
2.3 The Continuous Schwarz Algorithm

We consider the following problem. Find \( u \in H^1_0(\Omega) \) the solution of

\[
\begin{aligned}
&b(u, v - u) \geq (f + \lambda u, v - u) \\
&u \leq \Psi, v \leq \Psi
\end{aligned}
\]  

(15)

We decompose \( \Omega \) into two overlapping polygonal subdomains \( \Omega_1 \) and \( \Omega_2 \) such that

\[
\Omega = \Omega_1 \cup \Omega_2
\]

(16)

In [10], the solution \( u \) satisfies the condition of the following local regularity

\[
u_i / \Omega_i \in W^{2,p}(\Omega_i), \quad 2 \leq p < \infty
\]

(17)

We denote \( \partial \Omega_i \) the boundary of \( \Omega_i \) and

\[
\Lambda_1 = \partial \Omega_1 \cap \Omega_2, \Lambda_2 = \partial \Omega_2 \cap \Omega_1
\]

(18)

We assume that

\[
\Lambda_1 \cap \Lambda_2 = \emptyset
\]

(19)

2.3.1 A Continuous Algorithm

Starting from

\[
u_1^0 = 0, u_2^0 = \bar{u}
\]

(20)

such that \( \bar{u} \) is a solution of the following equation

\[
b(\bar{u}, v) = (f + \lambda \bar{u}, v), \quad \forall v \in K(\Psi, 0)
\]

(21)

We define the continuous sequence of Schwarz \( (u^n)_{n \in \mathbb{N}} \) such that

On \( \Omega_1 \), we have

\[
\begin{aligned}
&u_1^{n+1} \in K(\Psi, 0) \text{ is a solution of} \\
b_1(u_1^{n+1}, v - u_1^{n+1}) \geq (f_1 + \lambda u_1^n, v - u_1^{n+1}), \quad \forall v \in K(\Psi, 0) \\
u_1^{n+1} = u_2^n \text{ on } \Lambda_1, v = u_2^n \text{ on } \Lambda_1
\end{aligned}
\]

(22)

On \( \Omega_2 \), we have

\[
\begin{aligned}
&u_2^{n+1} \in K(\Psi, 0) \text{ is a solution of} \\
b_2(u_2^{n+1}, v - u_2^{n+1}) \geq (f_2 + \lambda u_2^n, v - u_2^{n+1}), \quad \forall v \in K(\Psi, 0) \\
u_2^{n+1} = u_1^{n+1} \text{ on } \Lambda_2, v = u_1^{n+1} \text{ on } \Lambda_2
\end{aligned}
\]

(23)

where \( f_i = (f + \lambda u^n) / \Omega_i, i = 1, 2 \).

and \( u_i = u / \Omega_i, i = 1, 2 \).

Theorem 2.5 ([12]) The sequences \( (u_1^{n+1}) \) and \( (u_2^{n+1}) \) ; \( n \geq 0 \) of the Schwarz method converge geometrically to the unique solution \( u \) of the obstacle problem (15), \( \exists \rho \in [0, 1[, \quad \forall n \geq 0

\[
\|u_i - u_i^{n+1}\|_{L^\infty(\Omega_i)} \leq (\rho)^n \|u_0 - u\|_{L^\infty(\Omega_i)}, i = 1, 2
\]

(24)
3 The Discrete Problem

3.1 Discretization

Let $V_{h_i} = V_{h_i} (\Omega_i)$ be the space of continuous piecewise linear functions on $\tau_{h_i}$ which vanish on $\partial \Omega \cap \Omega_i$. For $w \in C(\Lambda_i)$, we define the following space

$$V_{h_i}^{(w)} = \{ v \in V_{h_i} \mid v = 0 \text{ on } \partial \Omega \cap \Omega_i; v = \pi_{h_i}(w) \text{ on } \Lambda_i \}$$  \hspace{1cm} (25)

where $\pi_{h_i}$ denotes the interpolation operator on $\Lambda_i$. For $i = 1, 2$, let $\tau_{h_i}$ be a standard regular finite element triangulation in $\Omega_i$, $h_i$ being the meshsize. We suppose that the two triangulations are mutually independent on $\Omega_1 \cup \Omega_2$ A triangle belonging to one triangulation does not necessarily belong to the other. We assume that the corresponding matrices resulting from the discretizations of problems (22) and (23), are M-matrices ([15]).

3.2 Position of The Discrete Problem

The discrete problem is find $u_h \in H_0^1(\Omega)$ the solution of

$$\begin{cases} b(u_h, v_h - u_h) \geq (f + \lambda u_h, v_h - u_h) \\ u_h \leq r_h \Psi; v_h \leq r_h \Psi \end{cases}$$  \hspace{1cm} (26)

**Theorem 3.1** ([8]) Under the conditions in (5) to (11) and the maximum principle, there exists a constant $C$ independent of $h$ such that

$$\| u - u_h \|_{L^\infty(\Omega)} \leq Ch^2 |\ln h|^2$$  \hspace{1cm} (27)

3.3 The Discrete Schwarz Method

We give the discrete Schwarz method defined in (22) and (23) as follows.

3.3.1 Discrete Algorithm

Starting from

$$u_{0h}^0 = 0, u_{2h}^0 = \overline{u}_h$$ \hspace{1cm} (28)

such that $\overline{u}_h$ is a solution of the following equation

$$b(\overline{u}_h, v) = (f + \lambda \overline{u}_h, v), \forall v \in K_{(\Psi, 0)}$$ \hspace{1cm} (29)

We define the discrete sequence of Schwarz $(u_n^h)_{n \in \mathbb{N}}$ such that
Uniform convergence of Schwarz method for QVI

\[
\begin{align*}
&\left\{ \begin{array}{l}
u_{1h}^{n+1} \in V_{h1}^{(u_{n1h})} \text{ is a solution of} \\
b_1(u_{1h}^{n+1}, v - u_{1h}^{n+1}) \geq (f_1 + \lambda u_{1h}^{n}, v - u_{1h}^{n+1}), \forall v \in V_{h1}^{(u_{2h})} \\
u_{1h}^{n+1} \leq r_h \Psi, v \leq r_h \Psi 
\end{array} \right. \\
&\text{and} \\
&\left\{ \begin{array}{l}
u_{2h}^{n+1} \in V_{h2}^{(u_{n1h})} \text{ is a solution of} \\
b_2(u_{2h}^{n+1}, v - u_{2h}^{n+1}) \geq (f_2 + \lambda u_{2h}^{n}, v - u_{2h}^{n+1}), \forall v \in V_{h2}^{(u_{n1h})} \\
u_{2h}^{n+1} \leq r_h \Psi, v \leq r_h \Psi 
\end{array} \right.
\end{align*}
\]

(30)

Remark 3.2 Zhou in [16] gives the algebraic form of the discrete algorithm and the geometrical convergence of the sequence.

\[
\| u_{ih} - u_{ih}^{n+1} \|_{L^\infty(\Omega_i)} \leq (\theta^n) \| u_{ih} - u_{ih}^0 \|_{L^\infty(\Lambda_i)} \; ; \; 0 < \theta < 1; i = 1, 2. 
\]

(32)

4 \( L^\infty \)–Error Estimate

4.1 Auxiliary Sequences

We introduce two discrete auxiliary sequences. Starting from

\[
w_{1h}^0 = 0, w_{2h}^0 = \overline{u}_h
\]

(33)

such that \( \overline{u}_h \) is a solution of the following equation

\[
b(\overline{u}_h, v) = (f + \lambda \overline{u}_h, v), \forall v \in K_{(\Psi, \theta)} 
\]

(34)

We define the discrete auxiliary sequences \( (u_{h}^n)_{n \in \mathbb{N}} \) such that

\[
\left\{ \begin{array}{l}
u_{1h}^{n+1} \in V_{h1}^{(u_{n1h})} \text{ is the solution of} \\
b_1(u_{1h}^{n+1}, v - u_{1h}^{n+1}) \geq (f_1 + \lambda u_{1h}^{n}, v - u_{1h}^{n+1}), \forall v \in V_{h1}^{(u_{2h})} \\
u_{1h}^{n+1} \leq r_h \Psi, v \leq r_h \Psi 
\end{array} \right. \\
\text{and} \\
\left\{ \begin{array}{l}
u_{2h}^{n+1} \in V_{h2}^{(u_{n1h})} \text{ is the solution of} \\
b_2(u_{2h}^{n+1}, v - u_{2h}^{n+1}) \geq (f_2 + \lambda u_{2h}^{n}, v - u_{2h}^{n+1}), \forall v \in V_{h2}^{(u_{n1h})} \\
u_{2h}^{n+1} \leq r_h \Psi, v \leq r_h \Psi 
\end{array} \right.
\]

(35)

(36)

Denote by \( w_{ih}^{n+1} \) the finite element approximation of \( u_{ih}^{n+1} \) defined in (22) and (23).

We will give the lemma which plays a key role in proving our principal result in this paper.
We specify the following notations.

\[ \| \cdot \|_1 = \| \cdot \|_{L^\infty(\Lambda_1)}, \| \cdot \|_2 = \| \cdot \|_{L^\infty(\Lambda_2)} \]

\[ \| \cdot \|_1 = \| \cdot \|_{L^\infty(\Omega_1)}, \| \cdot \|_2 = \| \cdot \|_{L^\infty(\Omega_2)} \]

\[ \pi_{h_1} = \pi_{h_2} = \pi_h \]

The following lemma given in [1].

**Lemma 4.1** We have

\[
\| u_1^{n+1} - u_{1h}^{n+1} \|_{L^\infty(\Omega_1)} \leq \sum_{p=1}^{n+1} \| u_1^p - w_{1h}^p \|_{L^\infty(\Omega_1)} + \sum_{p=0}^{n} \| u_2^p - w_{2h}^p \|_{L^\infty(\Omega_2)} \tag{37}
\]

\[
\| u_2^{n+1} - u_{2h}^{n+1} \|_{L^\infty(\Omega_2)} \leq \sum_{p=1}^{n+1} \| u_2^p - w_{2h}^p \|_{L^\infty(\Omega_2)} + \sum_{p=1}^{n+1} \| u_1^p - w_{1h}^p \|_{L^\infty(\Omega_1)} \tag{38}
\]

**Proof.** The demonstration is an adaptation of the one in [1] given for the problem of variational inequality.

But our contribution is that one demonstrates that this lemma remained true for the problem introduced in this paper, while using a proposition with \( g + \lambda u \) and either \( g \) and \( \Psi \) as in [1].

To simplify the notation, one takes

\[ h_1 = h_2 = h \]

For \( n = 0 \), using the discrete form of proposition 2.4, we get

\[
\| u_1^1 - u_{1h}^1 \|_1 \leq \| u_1^1 - w_{1h}^1 \|_1 + \| w_{1h}^1 - u_{1h}^1 \|_1 \leq \| u_1^1 - w_{1h}^1 \|_1 + \| \pi_h u_2^0 - \pi_h u_{2h}^0 \|_1 \\
\hspace{1cm} \leq \| u_1^1 - w_{1h}^1 \|_1 + \| u_2^0 - u_{2h}^0 \|_1 \leq \| u_1^1 - w_{1h}^1 \|_1 + \| u_2^0 - u_{2h}^0 \|_2 \\
\| u_2^1 - u_{2h}^1 \|_2 \leq \| u_2^1 - w_{2h}^1 \|_2 + \| w_{2h}^1 - u_{2h}^1 \|_2 \leq \| u_2^1 - w_{2h}^1 \|_2 + \| \pi_h u_1^1 - \pi_h u_{1h}^1 \|_2 \\
\hspace{1cm} \leq \| u_2^1 - w_{2h}^1 \|_2 + \| u_1^1 - u_{1h}^1 \|_2 \leq \| u_2^1 - w_{2h}^1 \|_2 + \| u_1^1 - u_{1h}^1 \|_1 \\
\hspace{1cm} \leq \| u_2^1 - w_{2h}^1 \|_2 + \| u_1^1 - w_{1h}^1 \|_1 + \| u_2^0 - u_{2h}^0 \|_2 \\
\]

Therefore

\[
\| u_1^1 - u_{1h}^1 \|_1 \leq \sum_{p=1}^{1} \| u_1^p - w_{1h}^p \|_1 + \sum_{p=0}^{0} \| u_2^p - w_{2h}^p \|_{\Omega_2} \\
\| u_2^1 - u_{2h}^1 \|_2 \leq \sum_{p=0}^{1} \| u_2^p - w_{2h}^p \|_2 + \sum_{p=1}^{1} \| u_1^p - w_{1h}^p \|_1 \\
\]
For $n = 1$, using the discrete form of proposition 2.4, we get
\[
\|u^1_2 - u^{1h}_2\|_2 \leq \|u^1_2 - w^1_{2h}\|_2 + \|w^1_{2h} - u^{1h}_2\|_2 \leq \|u^2_1 - w^1_{1h}\|_1 + |\pi_h u^1_2 - \pi_h u^{1h}_2|_1 \\
\leq \|u^2_1 - w^1_{1h}\|_1 + |u^1_2 - u^{1h}_2|_1 \leq \|u^2_1 - w^1_{1h}\|_1 + \|u^1_2 - u^{1h}_2\|_2 \\
\leq \|u^2_1 - w^1_{1h}\|_1 + |u^1_2 - u^{1h}_2|_2 + \|u^1_1 - u^{1h}_1|_1 + \|u^0_2 - u^{0h}_2|_2 \\
\|u^2_2 - u^{2h}_2\|_2 \leq \|u^2_2 - w^2_{2h}\|_2 + \|w^2_{2h} - u^{2h}_2\|_2 \leq \|u^2_2 - w^2_{2h}\|_2 + |\pi_h u^1_2 - \pi_h u^{1h}_2|_2 \\
\leq \|u^2_2 - w^2_{2h}\|_2 + |u^1_2 - u^{1h}_2|_2 \leq \|u^2_2 - w^2_{2h}\|_2 + |u^1_2 - u^{2h}_2|_1 \\
\leq \|u^2_2 - w^2_{2h}\|_2 + \|u^1_2 - w^{2h}_1\|_1 + \|u^1_2 - w^{1h}_2\|_2 + \|u^1_1 - w^{1h}_1|_1 + \|u^0_2 - u^{0h}_2|_2 \\
\sum_{p=1}^2 \|u^p_2 - w^p_{1h}\|_1 \leq \sum_{p=1}^2 \|u^p_2 - w^p_{2h}\|_2 \\
\|u^2_2 - u^{2h}_2\|_2 \leq \sum_{p=0}^n \|u^p_2 - w^p_{2h}\|_2 + \sum_{p=1}^n \|u^p_1 - w^p_{1h}\|_1 \\
\|u^n_2 - u^{n2h}_2\|_2 \leq \sum_{p=0}^n \|u^p_2 - w^p_{2h}\|_2 + \sum_{p=1}^n \|u^p_1 - w^p_{1h}\|_1 \\
\|u^{n+1}_1 - u^{n+1h}_1\|_1 \leq \|u^{n+1}_1 - w^{n+1}_{1h}\|_1 + \|w^{n+1}_{1h} - u^{n+1}_1\|_1 \leq \|u^{n+1}_1 - w^{n+1}_{1h}\|_1 + |\pi_h u^1_2 - \pi_h u^{1h}_2|_1 \\
\leq \|u^{n+1}_1 - w^{n+1}_{1h}\|_1 + |u^0_2 - u^{n2h}_2|_1 \leq \|u^{n+1}_1 - w^{n+1}_{1h}\|_1 + \|u^0_2 - w^{n2h}_2\|_2 \\
\leq \|u^{n+1}_1 - u^{n2h}_2\|_1 + \sum_{p=0}^n \|u^p_2 - w^p_{2h}\|_2 + \sum_{p=1}^n \|u^p_1 - w^p_{1h}\|_1 \\
\|u^{n+1}_1 - u^{n+1h}_1\|_1 \|u^p_2 - w^p_{2h}\|_2 + \sum_{p=1}^n \|u^p_1 - w^p_{1h}\|_1 \\
\|u^{n+1}_1 - u^{n+1h}_1\|_1 \leq \|u^{n+1}_1 - w^{n+1}_{2h}\|_2 + \|w^{n+1}_{2h} - u^{n+1}_1\|_2 \leq \|u^{n+1}_1 - w^{n+1}_{2h}\|_2 + |\pi_h u^{n+1}_1 - \pi_h u^{n+1h}_1|_2 \\
\|u^{n+1}_2 - u^{n+1h}_2\|_2 \leq \|u^{n+1}_2 - w^{n+1}_{2h}\|_2 + \|w^{n+1}_{2h} - u^{n+1}_2\|_2 \leq \|u^{n+1}_2 - w^{n+1}_{2h}\|_2 + \|u^{n+1}_2 - u^{n+1h}_2\|_2 \\
\|u^{n+1}_2 - u^{n+1h}_2\|_2 \leq \|u^{n+1}_2 - w^{n+1}_{2h}\|_2 + \|w^{n+1}_{2h} - u^{n+1}_2\|_2 \leq \|u^{n+1}_2 - u^{n+1h}_2\|_2 + |u^{n+1}_2 - u^{n+1h}_2|_1
\[ \leq \|u_2^{n+1} - u_{2h}^{n+1}\|_2 + \sum_{p=0}^n \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 \]

Finally, we have
\[ \|u_2^{n+1} - u_{2h}^{n+1}\|_2 \leq \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 \]

4.2 $L^\infty$–Error Estimate

We finish by $L^\infty$–error estimate.

**Theorem 4.2** There exists a constant $C$ independent of both $h$ and $n$ such that
\[ \|u_i - u_{ih}\|_{L^\infty(\Omega_i)} \leq Ch^2 |\ln h|^3 ; i = 1, 2. \] (39)

**Proof.** For $i = 1$, we have
\[ \|u_1 - u_{1h}\|_1 \leq \|u_1 - u_1^{n+1}\|_1 + \|u_1^{n+1} - u_{1h}^{n+1}\|_1 + \|u_{1h}^{n+1} - u_{1h}\|_1 \]

We used theorem 2.5 and remark 3.2
\[ \leq (\rho)^{2n} |u^0 - u|_1 + \|u_1^{n+1} - u_{1h}^{n+1}\|_1 + (\theta)^{2n} |u_h - u_h^0|_1 \]

and lemma 4.1
\[ \leq (\rho)^{2n} |u^0 - u|_1 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^n \|u_2^p - w_{2h}^p\|_2 + (\theta)^{2n} |u_h - u_h^0|_1 \]

and theorem 3.1
\[ \leq (\rho)^{2n} |u^0 - u|_1 + 2(n + 1) Ch^2 |\ln h|^2 + (\theta)^{2n} |u_h - u_h^0|_1 \]

Let’s put $(\alpha)^{2n} \leq h^2$
where $\alpha = \max(\rho, \theta)$, therefore we find
\[ \|u_1 - u_{1h}\|_{L^\infty(\Omega_1)} \leq Ch^2 |\ln h|^3 \]

we get similar result for $i = 2$. 
References


Received: March, 2009