The Differential Formula for the First Derivative of Unit Timelike Split Quaternion Curve in the Semi-Euclidean Space $\mathbb{E}^4_2$

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Abstract
In this paper, a differential formula is given for the first derivative of non-null unit Split Quaternion Curve in the semi-Euclidean space $\mathbb{E}^4_2$

Keywords: Split quaternion, semi-Euclidean

1 Introduction

Shoemake introduced the unit quaternions to the computer graphics community for the purpose of controlling 3D solid rotations [5]. M.J Kim, M.S Kim and S.Y Shin derived a compact differential formula for the first derivative of a unit quaternion curve defined on $SO(3)$ or $S^3$. They denoted the effectiveness of this formula by deriving the differential properties of various unit quaternion curves [3]. We give some definitions and theorems relation semi-Euclidean space and split quaternion.

$\mathbb{E}^n$ with the metric tensor $\langle u, v \rangle |_L = -\sum_{i=1}^{\nu} v_i w_i + \sum_{j=\nu+1}^{n} v_j w_j$ $v, w \in \mathbb{E}^n$, $0 \leq \nu \leq n$ is called the semi-Euclidean space and is shown by $\mathbb{E}^n_{\nu}$ where $\nu$ is called the index of the metric [1].
The algebra of unit split quaternions is denoted by $\mathbb{H}$ and defined in the semi-Euclidean space $\mathbb{E}^4_{2}$. An element of $\mathbb{H}$ is called as unit split quaternion. A unit split quaternion $h$ is expressed of the form $h = h_0 + h_1e_1 + h_2e_2 + h_3e_3$ where $h_0, h_1, h_2$ and $h_3$ are real numbers, and $e_1, e_2, e_3$ are split quaternionic units [2].

**Theorem 1.1** Every unit spacelike split quaternion can be expressed $h = (\sinh \gamma + w \cosh \gamma)$ whose vectorial part is spacelike unit vector $\varepsilon_0$. Every unit timelike split quaternion can be expressed $h = (\cosh \gamma + w \sinh \gamma)$ whose vectorial part is spacelike unit vector $w$ such that $h = v \ast u^{-1}$. $\gamma$ is the hyperbolic angle between Lorentzian vectors $u$ and $v$ satisfying one of conditions.

In this case unit timelike quaternion $h = (\cosh \gamma + w \sinh \gamma)$ respectively correspond to a great Lorentzian circle of $H^2_0^+$ (or $H^2_0^-$) and a great Lorentzian circle of $S^2_1$ [4].

**Theorem 1.2.** Let $q = \cosh \alpha + w \sinh \alpha$ be a timelike quaternion with spacelike vector part and $\varepsilon$ be a Lorentzian vector. Then the transformation $R_q(\varepsilon) = q \ast \varepsilon \ast q^{-1}$ is a rotation through hyperbolic angle $2\alpha$ about the spacelike axis $w$ [4].

## 2 Differential Formula For Unit Timelike Split Quaternions with spacelike vector part

Each solid orientation obtained is the same as that obtained by rotating the solid about the unit spacelike axis $\sqrt{\frac{h_1^2+h_2j+h_3k}{-h_1^2+h_2^2+h_3^2}}$ by an angle hyperbolic $2\gamma t$, $0 \leq t \leq 1$, and the representation corresponding quaternion of the rotation is denoted as following $\psi(t) = \left( \cosh \gamma t, \frac{h_1i+h_2j+h_3k}{\sqrt{-h_1^2+h_2^2+h_3^2}} \sinh \gamma t \right)$.

$\psi(0) = \left( 1, 0, \frac{h_1i+h_2j+h_3k}{\sqrt{-h_1^2+h_2^2+h_3^2}} \right) = (1, 0, 0, 0)$,

$\psi(1) = \left( \cosh \gamma, \frac{h_1i+h_2j+h_3k}{\sqrt{-h_1^2+h_2^2+h_3^2}} \sinh \gamma \right) = p_2 \ast p_4^{-1} \in S^3_2$. 
\[ \psi'(t) = \left( \gamma \sinh \gamma t, \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \gamma \cosh \gamma t \right) \in E_2^4 \]

\[ \psi'(0) = \left( 0, \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \right) \in E_2^4, \]

The path \( \psi(t) = \cosh \gamma t (1, 0, 0, 0) + \sinh \gamma t \left( 0, \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \right), \quad (0 \leq t \leq 1) \) is on a great Lorentzian circle arc \( H_0^{2+} \) (or \( H_0^{2-} \)) or a great Lorentzian circle arc \( S^2_1 \). \( \psi(t) = \left( \gamma \sinh \gamma t, \gamma \cosh \gamma t \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \right), \quad 0 \leq t \leq 1, \) is a geodesic arc between the two unit timelike \( 1 \) and \( p_2 * p_1^{-1} \in S^3_2 \). A natural generalization of Euler’s formula for timelike split quaternions with unit spacelike split quaternion axis \( \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \in \mathbb{E}_4^3 \) is

\[
e^{(0, \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}})} = 1 + \left( \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \right) \gamma + \left( \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \right)^2 \frac{\gamma^2}{2!} + \cdots
\]

Thus, the logaritm for a unit spacelike quaternion \( \cosh \gamma + \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \sinh \gamma \in S^3_2 \) is given by

\[
\log \left( \cosh \gamma + \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \sinh \gamma \right) = \left( 0, \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \gamma \right).
\]

The geodesic path \( \psi(t) \in S^3_2 \), for \( 0 \leq t \leq 1 \), between two unit timelike quaternions \((1, 0, 0, 0)\) and \( p_2 * p_1^{-1} = \left( \cosh \gamma, \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \sinh \gamma \right) \in S^3_2 \) can be represented by
\[
\psi(t) = \left( \cosh \gamma t, \sinh \gamma t \left( 0, \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}} \right) \right) = e^{(0, t \gamma, \frac{h_1 i + h_2 j + h_3 k}{\sqrt{-h_1^2 + h_2^2 + h_3^2}})} = e^{(t \log(p_2 * p_1^{-1}))}.
\]

The geodesic circular arc \( \psi_{p_1, p_2}(t) \in S^3_2 \), which connects two unit timelike split quaternions \( p_1 \) and \( p_2 \in S^3_2 \) is given by \( \psi_{p_1, p_2}(t) = \psi(t) * p_1 = e^{(t \log(p_2 * p_1^{-1}))} * p_1 \)

The curve differential is also given by

\[
\psi_{p_1, p_2}'(t) = \log(p_2 * p_1^{-1}) \psi_{p_1, p_2}(t)
\]

**Definition 2.1.** The exponential map \( e^{(h_1, h_2, h_3)} : \mathbb{E}^3_1 \to S^3_2 \subset \mathbb{E}^4_2 \) can be defined as follows

\[
\begin{align*}
e^{(h_1, h_2, h_3)} &= \begin{cases} 
\text{cosh} \sqrt{\langle v, v \rangle}, \frac{\sinh \sqrt{\langle v, v \rangle}}{\sqrt{\langle v, v \rangle}} (h_1, h_2, h_3) & \text{if } v = (h_1, h_2, h_3) \neq (0, 0, 0) \\
(1, 0, 0, 0) & \text{if } v = (h_1, h_2, h_3) = (0, 0, 0)
\end{cases}
\end{align*}
\]

**Theorem 2.1.** For unit timelike split quaternion \( h = (h_0, h_1, h_2, h_3) \in S^3_2 \) with unit spacelike vector part \((h_1, h_2, h_3) \neq (0, 0, 0) \in \mathbb{E}^3_1\) the differential \( de^{(h_1, h_2, h_3)} \) can be denoted as follows

\[
\left[ -\frac{M}{\langle v, v \rangle} h_1 + \frac{N}{\langle v, v \rangle^2} h_1^2 + \frac{N}{\langle v, v \rangle} h_2 h_1, -\frac{M}{\langle v, v \rangle} h_2 + \frac{N}{\langle v, v \rangle^2} h_2^2 + \frac{N}{\langle v, v \rangle} h_3 h_2, -\frac{M}{\langle v, v \rangle} h_3 h_2, -\frac{M}{\langle v, v \rangle} h_3^2 + \frac{N}{\langle v, v \rangle} h_3 \right]
\]

where \( M = \cosh \sqrt{\langle v, v \rangle} \) and \( N = \sinh \sqrt{\langle v, v \rangle} \)

**Theorem 2.2.** A map \( e^{(h_1, h_2, h_3)} : \mathbb{E}^3_1 \to S^3_2 \subset \mathbb{E}^4_2 \) is differentiable at \( H = (h_1, h_2, h_3) \in \mathbb{E}^3_1 \) which approaches to the origin \((0, 0, 0)\) if and only if there is a linear map \( L : \mathbb{E}^3_1 \to S^3_2 \subset \mathbb{E}^4_2 \) such that
The differential formula for the first derivative

\[ \lim_{H \to (0,0,0)} \frac{\|e^{(h_1, h_2, h_3)} - e^{(0,0,0)} - L(h_1, h_2, h_3)\|}{\|H\|} = \lim_{H \to (0,0,0)} \frac{\left(\cosh \sqrt{|\langle H, H \rangle|}, \sinh \sqrt{|\langle H, H \rangle|}, -1,0,0\right) - (1,0,0) - (0,1,0)}{\sqrt{|\langle H, H \rangle|}} = \lim_{H \to (0,0,0)} \left\| \left( \frac{(H,H)}{2t} + \frac{(H,H)^2}{4t^2} + \ldots, \frac{(H,H)^3}{6t^3} + \frac{(H,H)^4}{8t^4} + \ldots \right) (h_1, h_2, h_3) \right\| = 0 \]

The differential formula for unit timelike split quaternion with spacelike vector part \( h(t) \in S^3_2 \) is given by decomposing the unit timelike split quaternion with spacelike vector part curve \( h(t) \) into two maps, exp and log. A unit timelike split quaternion is denoted \( h(t) = e^{\log(p_2 * p_1^{-1})(t)} \). We have \( k(t) = \log(p_2 * p_1^{-1})(t) = (h_1, h_2, h_3) \in E_3^1 \) \( 0 \leq t \leq 1 \). Thus, we obtain \( h(t) = e^{\log(p_2 * p_1^{-1})(t)} = e^{k(t)} \). For the unit timelike split quaternion curve with spacelike vector part \( h(t) \in S^3_2 \), the differential \( h'(t) \) at \( t \in [0,1] \) such that \( k(t) = (0,0,0) \) can be represented \( h'(t) = de^{(0,0,0)}(k'(t)) = k'(t) = (h'_1, h'_2, h'_3) \in E_3^1 \)

References


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