Decomposition Properties of Sober Topologies on the Same Sets\textsuperscript{1}

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Abstract

In this paper, decomposition properties of sober topologies on the same sets are concerned. Main results are: (1) The supremum of a family of sober topologies on the same set with the same closures on every singletons remains a sober topology; (2) The supremum of a family of $T_1$ and sober topologies on the same set is still a $T_1$ and sober topology; (3) The supremum of a directed family of sober topologies on the same set remains a sober topology. Some subtle (counter)examples are also constructed to show that the supremum of two sober topologies on the same set may not be a sober one.

Keywords: topological space; sober topology; decomposition; supremum

1 Introduction

Topology (cf. [1]), one of the most important subjects in mathematics, provides mathematical tools and interesting topics in various subjects. Soberity, a special separation property of topological spaces, plays an important role in studying continuous lattices and domains (cf. [2, 3, 4]). Some properties for soberity have been established. It is known that every continuous domain with

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the Scott topology (an intrinsic topology of posets) is a sober space. It has been proved that finite products, saturated subspaces, closed subspaces and retracts of sober spaces are all sober spaces. Since the family of all topologies on a set forms a complete lattice, a question naturally arises: are the sups of sober topologies on the same sets are all still sober ones? The answer to this question is generally negative. This paper will carry on deeper and detailed studies on this topic. We will show the following theorems: (1) The supremum of a family of sober topologies on the same set with the same closures on every singletons remains a sober topology; (2) The supremum of a family of $T_1$ and sober topologies on the same set is still a $T_1$ and sober topology; (3) The supremum of a directed family of sober topologies on the same set remains a sober topology. In terms of the usual topology of $\mathbb{R}$, a counterexample of $T_1+$sober topological space which is not $T_2$ is constructed, showing that $T_2$ is strictly stronger than $T_1+$sober. We will also construct an example to show that the supremum of two sober topologies on the same set may not be a sober one.

The rest of this paper is organized as follows. Section 2 introduces fundamental concepts and presents basic facts of topological spaces. Section 3 gives some decomposition properties of sober topologies on the same sets. In section 4, some subtle and important (counter)examples are constructed.

## 2 Preliminaries

We recall notations and facts related to topological spaces and posets that are used in the sequel. Other used but not stated basic concepts please refer to [1, 2].

Let $(X, \tau)$ be a topological space. The closure of a subset $A$ of $X$ is denoted by $\overline{A}$ or $A^-$. When a family of topologies $\tau_i$ ($i \in J$) on the same set $X$ is concerned, we will use $\overline{A}^i$ to denote the closure of $A$ in the topological space $(X, \tau_i)$ ($i \in J$). A topological space $X$ is a $T_0$-space is characterized by that $x \neq y$ implies $\{x\}^- \neq \{y\}^-$ for any $x, y \in X$. A topological space is a $T_1$ space iff every of its singletons is closed. A $T_2$ space means a topological space in which any pair of distinct elements have disjoint open neighborhoods. It is well-known that for topological spaces, $T_2$ implies $T_1$ and $T_1$ implies $T_0$.

**Definition 2.1.** (cf. [2]) Let $(X, \tau)$ be a topological space. A non-empty subset $A$ of $X$ is called irreducible if $A \subseteq B \cup C$ for all closed sets $B$ and $C$ implies $A \subseteq B$ or $A \subseteq C$. 


Lemma 2.2. (cf. [2]) A non-empty subset $A$ of a topological space $X$ is irreducible iff its closure $\overline{A}$ is irreducible.

Definition 2.3. (cf. [2]) A topological space $X$ is sober if for every irreducible closed set $C$ there is a unique element $x \in X$ such that $\{x\} = C$.

It is easy to see that $\forall x \in X, \overline{\{x\}}$ is a irreducible closed set, and thus every sober space must be a $T_0$ space.

Proposition 2.4. Let $(X, \tau_1)$ and $(X, \tau_2)$ be topological spaces on the same set $X$. Let $\tau_1 \subseteq \tau_2$ and $A \subseteq X$. Then

1. $\overline{A}^2 \subseteq \overline{A}^1$.
2. If $A$ is irreducible in $(X, \tau_2)$, then $A$ is also irreducible in $(X, \tau_1)$.

Proof. Straightforward. \qed

Definition 2.5. Let $P$ be a poset. Then

(i) A non-empty subset $D$ of $P$ is called directed, if every pair of elements of $D$ has an upper bound in $D$.

(ii) Poset $P$ is called a dcpo if every its directed subset has the least upper bound.

(iii) A subset $U$ of $P$ is called Scott open if $U$ is an upper set and for all directed subset $D$ of $P$ with existing $\sup D \in U$, one has $D \cap U \neq \emptyset$. All the Scott open sets of $P$ form a topology, called the Scott topology and denoted by $\sigma(P)$.

Definition 2.6. (cf. [2]) The specialization order $\leq_s$ on a topological space $X$ is defined for all $x, y \in X$, $x \leq_s y$ iff $x \in \overline{\{y\}}$.

Remark 2.7. Let $(X, \tau)$ be a topological space. Then

(i) The specialization order $\leq_s$ is a preorder on $X$. If $(X, \tau)$ is a $T_0$ space, then $\leq_s$ is a partial order on $X$. If $(X, \tau)$ is a sober space, then $(X, \leq_s)$ is a dcpo.

(ii) It follows from the definition of the specialization order that for all $y \in X$, $\{y\}^- = \downarrow y$.

3 Decomposition Theorems of Sober Topologies

Recall that a topology of a sober space is called a sober topology.
Proposition 3.1. Let $\tau_i$ ($i \in J$) be a family of sober topologies on the same set $X$. If $\forall x \in X$ and $i \in J$, $\bar{x} = \bar{x}^i$, then the supremum topology $\tau = \bigvee \tau_i$ generated by the subbase $\bigcup \tau_i$ is a sober topology on $X$, where $\bar{x}^i$ is the closure of $\{x\}$ in space $(X, \tau_i)$.

Proof. Let $F$ be an irreducible closed set in $(X, \tau)$. Then $F$ is an irreducible set in $(X, \tau_i)$ $(i \in J)$. By Lemma 2.2, $F^i$ is an irreducible closed set in $(X, \tau_i)$ $(i \in J)$. Since $(X, \tau_i)$ is sober, there is a unique $x_i$ such that $F^i = \{x_i\}^i$ $(i \in J)$. By the assumption that $\bigcap_{i} x_i = \{x\}$ and $F \subseteq F^i = \{x_i\}$, we have $F^k \subseteq \{x_i\} = \{x\} = F^i$ $(i, k \in J)$. This shows that $F^i = F^k = \{x_i\} = \{x_k\}$ $(i, k \in J)$. Thus the soberity of $(X, \tau_i)$ $(i \in J)$, we have a unique $x = x_i = x_k \in X$ $(i, k \in J)$. Next we show that $\{x\} = F$. On one hand, it follows from $\tau_i \subseteq \tau$ that $x \in \{x\} \subseteq \{x\} = F^i$. Since $F$ is an irreducible closed set in $(X, \tau)$, $F$ can be represented as $F = \bigcap_{i} F_i$ for some closed set $F_i$ in $(X, \tau_i)$ $(i \in J)$. It follows from $F \subseteq F_i$ that $\{x\} = \{x\} = F^i \subseteq F_k$ $(i, k \in J)$. Thus $\{x\} \subseteq \{x\} \subseteq \bigcap_{i} F_i$. On the other hand, since $\{x\}$ is closed and irreducible in $(X, \tau)$, $\{x\}$ can be represented as $\{x\} = \bigcap_{i} C_i$ for some closed set $C_i$ in $(X, \tau_i)$ $(i \in J)$. Then it follows from $x \in F \subseteq F^i = \{x\} \subseteq C_i$ $(i \in J)$ that $F \subseteq \bigcap_{i} C_i = \{x\}$. To sum up the above, and noticing that $\tau$ is clearly $T_0$, we have that there is a unique $x \in X$ such that $\{x\} = F$. So, $\tau$ is a sober topology on $X$.

Corollary 3.2. Let $\tau_i$ $(i \in J)$ be a family of $T_1$-sober topologies on the same set $X$. Then the supremum topology $\tau = \bigvee \tau_i$ generated by the subbase $\bigcup \tau_i$ is a $T_1$-sober topology on $X$.

Proof. It is clear that $\tau$ is $T_1$. Since topologies $\tau_i$ $(i \in J)$ are $T_1$, $\forall x \in X$ and $i \in J$, $\bar{x} = \{x\} = \bar{x}^i$. Thus, by Proposition 3.1, $(X, \tau)$ is also sober.

Lemma 3.3. (cf. [5]) A topological space is a $T_1$-sober space iff its irreducible closed sets are all singletons.

Proof. Straightforward.

Proposition 3.4. Let $\tau$ be a $T_1$-sober topology and $\tau^* \supseteq \tau$ be a topology on the same set $X$. Then $\tau^*$ is a sober topology.

Proof. Topology $\tau^*$ is $T_1$ is clear. By Lemma 3.3, we need to show that irreducible closed sets are singletons. Let $F$ be an irreducible closed set in $(X, \tau^*)$. Then by Proposition 2.4, $F$ is an irreducible set in $(X, \tau)$. By Lemma 2.2, we have that the closure of $F$ in $(X, \tau)$ is an irreducible closed set. Thus by Lemma 3.3, (the closure of) $F$ is a singleton.
Proposition 3.5. Let $\tau_i$ $(i \in J)$ be a directed family of sober topologies on the same set $X$. Then the supremum topology $\tau = \bigvee \tau_i$ generated by the subbase $\bigcup \tau_i$ is a sober topology.

Proof. Let $F$ be an irreducible closed set in $(X, \tau)$. Then $F$ is irreducible in $(X, \tau_i)$ $(i \in J)$. By Lemma 2.2, $\mathcal{F}$ is an irreducible closed set in $(X, \tau_i)$ $(i \in J)$. Since $(X, \tau_i)$ is sober, there is a unique $x_i$ such that $\mathcal{F}_i = \{x_i\}$ $(i \in J)$. By the directedness of the family $\tau_i$ $(i \in J)$, we have that for all $i, j \in J$, there is $k \in J$ such that $\tau_i, \tau_j \subseteq \tau_k$. Thus, $F \subseteq \mathcal{F}_k = \{x_k\}$ and $\{x_i\} \subseteq \mathcal{F}_k$. By Proposition 2.4, we have that $x_k \in \{x_k\} = \mathcal{F}_k \subseteq \mathcal{F}$ and $\{x_i\} \subseteq \mathcal{F}$. These show that $\mathcal{F}_i = \{x_i\} = \{x_k\}$ and similarly, $\mathcal{F}_j = \{x_j\} = \{x_k\}$. Thus by the soberness of $(X, \tau_i)$ $(i \in J)$, we have $x_i = x_j$ $(\forall i, j \in J)$. Let $x = x_i = x_j$ $(i, j \in J)$. Next we show that $\{x\} = \mathcal{F}$. On one hand, it follows from $\tau_i \subseteq \tau$ and Proposition 2.4 that $x \in \{x\} = \mathcal{F}$. Since $F$ is an irreducible closed set in $(X, \tau)$, $F$ can be represented as $F = \bigcap_{i \in J} F_i$ for some closed set $F_i$ in $(X, \tau_i)$ $(i \in J)$. It follows from $F \subseteq F_i$ and $x \in \{x\} = \mathcal{F}_i \subseteq F_i$ $(i \in J)$ that $x \in \bigcap_{i \in J} F_i = F$. Thus $\{x\} = \mathcal{F}$. On the other hand, since $\{x\} = \mathcal{F}$ is closed and irreducible in $(X, \tau)$, $\{x\}$ can be represented as $\{x\} = \bigcap_{i \in J} C_i$ for some closed set $C_i$ in $(X, \tau_i)$ $(i \in J)$. Then it follows from $x \in F \subseteq \mathcal{F}_i = \{x\} \subseteq C_i$ $(i \in J)$ that $F \subseteq \bigcap_{i \in J} C_i = \{x\}$. To sum up the above, and noticing that $\tau$ is clearly $T_0$, we have that there is a unique $x \in X$ such that $\{x\} = \mathcal{F}$. So, $\tau$ is a sober topology on $X$. \qed

Corollary 3.6. Let $\tau_i$ $(i \in J)$ be a directed family of sober topologies on the same set $X$. If $\tau = \bigcup \tau_i$ is a topology, then $\tau$ is a sober topology on $X$.

Proof. If $\tau = \bigcup \tau_i$ is a topology, then $\bigcup \tau_i$ is equal to the supremum topology $\bigvee \tau_i$ generated by the subbase $\bigcup \tau_i$. By Proposition 3.5, $(X, \tau)$ is sober. \qed

4 Some examples

The first example shows that supremum of two sober topologies on the same set may fail to be a sober one, showing that the directedness in Proposition 3.5 can not be removed.

Example 4.1. Let $P$ be the poset of natures augmented two maximal elements $a, b$ to the top. Then the Scott topology $\sigma(P)$ on $P$ is not sober. Let $\tau_a = \sigma(P) - \{\{a\}, \{a, b\}\}$. Then $\tau_a$ is a sober topology on $P$ and so is topology $\tau_b = \sigma(P) - \{\{b\}, \{a, b\}\}$. It is easy to see that $\sigma(P) = \tau_a \vee \tau_b$ is not sober.
Next example shows that some union of a directed family of sober topologies on the same set $X$ may fail to be a topology, showing that the assumption of Corollary 3.6 is needed.

**Example 4.2.** Let $X = [0, 1]$. For all $a \in [0, 1)$, let $\tau_a$ be the topology generated by base $\mathcal{P}([0, a]) \cup \sigma((a, 1])$. Noticing that $(0, 1]$ is a continuous domain, we have that $\sigma((a, 1])$ is a sober topology on $(a, 1]$. Then it is easy to show that $\tau_a$ is a sober topology on $[0, 1]$. If $a \leq b$, then $\tau_a \subseteq \tau_b$. So the family $\tau_a$ ($a \in [0, 1)$) is a directed family of sober topologies on $X$. It is easy to see that $[0, 1)$ is not $\tau_a$-open for all $a \in [0, 1)$. However $[0, 1)$ is open in $(X, \bigvee_{a \in [0, 1)} \tau_a)$. This deduces that $\bigcup_{a \in [0, 1)} \tau_a$ is not a topology.

Every $T_2$ spaces is a $T_1+$sober space. And $T_1+$sober spaces share almost all known topological properties that $T_2$ spaces have (cf. [5]). However, the following example shows that $T_2$ is strictly stronger than $T_1+$sober.

**Example 4.3.** Let $(\mathbb{R}, \tau)$ be the real number space. Let $\tau^* = \{U \in \tau \mid U$ is a dense subset in $(\mathbb{R}, \tau)\} \cup \{\emptyset\}$. Then $(\mathbb{R}, \tau^*)$ is $T_1+$sober but not $T_2$.

It is a routing work to show that $\tau^*$ is a topology on $\mathbb{R}$. And it is clear that $\tau^*$ is $T_1$ but not $T_2$. To show $(\mathbb{R}, \tau^*)$ is $T_1+$sober, by Lemma 3.3, we need only to show irreducible closed sets in $(\mathbb{R}, \tau^*)$ are all singletons. Suppose that $F$ is an irreducible closed sets in $(\mathbb{R}, \tau^*)$ with more than two distinct elements $x, y \in F$ and $x < y$. It is clear by irreducibility of $F$ that $F \neq \mathbb{R}$. Since $\mathbb{R} - F$ is dense and the interval $(x, y]$ is open in $(\mathbb{R}, \tau)$, there is some $z \in (x, y] \cap (\mathbb{R} - F)$. Then $F_1 = (-\infty, z] \cap F$ and $F_2 = [z, \infty) \cap F$ are two closed subsets in $(\mathbb{R}, \tau^*)$ and $F = F_1 \cup F_2$, but $F_1 \neq F \neq F_2$, a contradiction! This contradiction reveals that irreducible closed sets in $(\mathbb{R}, \tau^*)$ are all singletons. So, by Lemma 3.3, $(\mathbb{R}, \tau^*)$ is a $T_1+$sober space but not a $T_2$ space.

**References**


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