Notes on Maximal Ideals
Relative to a Filter

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Abstract
In this paper, an intrinsic characterization of distributive lattices is obtained. In addition, we also give a characterization of pseudo primes in semicontinuous lattices.

Mathematics Subject Classification: 06A11; 06B35; 54H10

Keywords: maximal ideals relative to a filter, prime ideals, semiprime ideals, distributive lattices, semicontinuous lattices

1 Introduction and Preliminaries

The study of semiprime ideals was begun in [5] by Y. Ray. The theory of semicontinuous lattices was first developed by D. Zhao in [1]. In this paper, we manage to give an intrinsic characterizations of distributive lattices. A characterization of pseudo primes in semicontinuous lattices is also obtained.

The following are some basic concepts needed in the sequel, other non-explicitly stated elementary notions please refer to [1], [3] and [6].

An ideal on a partially ordered set (in short, poset) \( L \) means a lower set which is also directed, and a filter on a poset can be dually defined. For a semilattice \( L \), a proper ideal \( I \) of \( L \) is called a prime ideal if for any two elements \( a, b \) of \( L \), \( a \land b \in I \) implies \( a \in I \) or \( b \in I \). For a lattice \( L \), an ideal \( I \) of \( L \) is called a semiprime ideal if for any three elements \( a, b, c \) of \( L \), the relations \( a \land b \in I, a \land c \in I \) always imply \( a \land (b \lor c) \in I \). The set of semiprime
ideals of $L$ is denoted by $Rd(L)$. It is easy to see that every prime ideal is semiprime in a lattice.

Recall that in a complete lattice $L$, for $x, y \in L$, we say that $x \preceq y$, if for any $I \in Rd(L)$, $y \leq \bigvee I$ always implies $x \in I$. For any $x \in L$, let $\downarrow x = \{ y \in L : y \preceq x \}$ and $\uparrow x = \{ y \in L : x \preceq y \}$. A complete lattice $L$ is said to be semicontinuous lattice if for each $x \in L$, $x \leq \bigvee \downarrow x$.

2 Main Results

In this section, we shall give an intrinsic characterization of distributive lattices. A characterization of pseudo primes in the case of semicontinuous lattices will be obtained.

**Lemma 2.1.** (see [4]) Let $L$ be a sup-semilattice. Let $I \in Idl L$, $F \in Filt L$ and $I \cap F = \emptyset$. Then $I$ is a maximal ideal relative to filter $F$ iff for all $x \in L \setminus I$, there are $y \in F$ and $a \in I$ s.t. $y \leq x \lor a$.

**Theorem 2.2.** Let $M$ be a semiprime ideal on a lattice. If $M$ is a maximal ideal relative to a filter, then $M$ is a prime ideal.

**Proof.** Let $L$ be a lattice. Let $M$ be a maximal ideal relative to filter $F$ and $M \cap F = \emptyset$. Suppose there are $a, b \in M$ s.t. $a \land b \in M$ but $a \not\in M$ and $b \not\in M$. By Lemma 2.1, there are $u, v \in F$, $c, d \in M$ s.t. $u \leq a \lor c$, $v \leq b \lor d$, respectively. Since $F$ is a filter, $a \lor c$, $b \lor d \in F$ and $(a \lor c) \land (b \lor d) \in F$.

Noticing that $c, d \in M$ and $M$ is a semiprime ideal, we have $a \land b \in M$, $a \land d \in M$ and $a \land (b \lor d) \in M$; and also $c \land b \in M$, $c \land d \in M$ and $c \land (b \lor d) \in M$. It follows from $M$ is a semiprime ideal that $(a \lor c) \land (b \lor d) \in M$. This shows that $(a \lor c) \land (b \lor d) \in M \cap F \neq \emptyset$, a contradiction! Hence, $M$ is a prime ideal. \qed

This theorem shows that the similar result may be obtained in the non-distributive case. An example is given by Figure 1 to show that maximal ideals relative to a filter may be prime ideals in the non-distributive case, where $I = \{ a, b, c, d, 0 \}$ is a maximal ideal relative to filter $L \setminus I$ and a prime ideal but $L$ is not a distributive lattice.

By Figure 1, we find that there exists a maximal ideal $\downarrow b$ relative to filter $\uparrow a$ but not a semiprime ideal in the non-distributive lattice $L$.

**Lemma 2.3.** (see [4]) Let $L$ be a poset, $I \in Idl L$, $F \in Filt L$ and $I \cap F = \emptyset$. Then there always exists a maximal ideal $M$ relative to filter $F$ s.t. $M \cap F = \emptyset$ and $M \supseteq I$. 

![Figure 1](image-url)
Theorem 2.4. If maximal ideals relative to a filter on a lattice $L$ are all semiprime ideals, then $L$ is a distributive lattice.

Proof. Let $L$ be a lattice and $a, b, c \in L$. Let $x = a \wedge (b \vee c)$ and $y = (a \wedge b) \vee (a \wedge c)$. It is trivial that $x \geq y$. On the other hand, suppose that $x \not\leq y$. Then $\uparrow x \cap \downarrow y = \emptyset$. By Lemma 2.3, there exists a maximal ideal $M$ relative to filter $\uparrow x$ s.t. $M \cap \uparrow x = \emptyset$ and $M \supseteq \downarrow y$. Thus $x \notin M$ and $y \in M$. Since $M$ is a semiprime ideal, $a \wedge b \in M, a \wedge c \in M$ and $x = a \wedge (b \vee c) \in M$, a contradiction! Hence $x \leq y$, and thus $L$ is distributive.

Corollary 2.5. If maximal ideals on a lattice $L$ are all semiprime ideals, then $L$ is a distributive lattice.

By Theorem 2.4, Corollary 2.5 and the fact that maximal ideals relative to a filter on a distributive lattice are all prime ideals (see [4]), we have an intrinsic characterization of distributive lattices.

Theorem 2.6. Let $L$ be a lattice. Then the following conditions are equivalent:

1. $L$ is a distributive lattice;
2. Maximal ideals relative to a filter on $L$ are all prime ideals;
3. Maximal ideals relative to a filter on $L$ are all semiprime ideals;
4. Maximal ideals on $L$ are all semiprime ideals;
5. Maximal ideals on $L$ are all prime ideals.

Recall that an element $p$ of a poset $L$ is called pseudo prime element if $p = \bigvee P$ for some prime ideal $P$. All the pseudo prime elements of $L$ is denoted by $\psi_{\text{PRIME}} L$.

Now we give the following characterization of pseudo primes in semicontinuous lattices.

Lemma 2.7. ([3]) Let $L$ be a distributive lattice, $I$ an ideal and $F$ a filter in $L$ with $I \cap F = \emptyset$. Then there is a prime ideal $P$ in $L$ with $P \supseteq I$ and $P \cap F = \emptyset$.

Theorem 2.8. Let $L$ be a complete lattice and $1 \neq p \in L$. Consider the following statements:

1. $p$ is pseudo prime;
2. In any finite collection $x_1, x_2, \cdots, x_n \in L$ with $x_1 \wedge x_2 \wedge \cdots \wedge x_n \leq p$ there is one of the elements with $x_j \leq p$;
3. The filter generated by $L \backslash \downarrow p$ does not meet $\downarrow p$.

Then (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3); if $L$ is in addition distributive semicontinuous, all three statements are equivalent.
Proof. Condition (2) says that no finite meet of elements from $L \downarrow p$ is ever $\ll p$. Therefore (2) and (3) are always equivalent.

(1) implies (2): Let $p$ be pseudo prime and suppose that $x_1 \land x_2 \land \cdots \land x_n \ll p$. Let $P$ be a prime ideal with $\bigvee P = p$. Since every prime ideal is semiprime, $P \in Rd(L)$, thus $x_1 \cdots x_n \in P$. Since $P$ is prime, there is one $j \in \{1, 2, \cdots, n\}$ with $x_j \in P \subseteq \downarrow p$. That is, $x_j \leq p$.

(3) implies (1): Suppose that $L$ is semicontinuous. Let $F$ be the filter generated by $L \downarrow p$, then $L \downarrow p \subseteq F$ and $F \cap \downarrow p = \emptyset$. By Lemma 2.7, there exists a prime ideal $P$ with $P \supseteq \downarrow p$ and $P \cap F = \emptyset$. Since that $L \downarrow p \subseteq F$, we have $P \subseteq L \downarrow p \subseteq \downarrow p$. Since $L$ is semicontinuous, $p \leq \bigvee \downarrow p \leq \bigvee P \leq \bigvee \downarrow p = p$. Thus $p = \bigvee P$ is pseudo prime.

References


Received: January, 2009