On the Third Order Rational Difference Equation

\[ x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})} \]

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Abstract

In this paper we will give the solutions of the rational difference equation \( x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})} \), \( n = 0, 1, \ldots \) where initial values \( x_0, x_{-1}, \) and \( x_{-2} \) are nonnegative real numbers with \( bx_0 x_{-2} \neq -a \) and \( x_{-1} \neq 0 \). Moreover we investigate some properties for this difference equation such as the local stability and the boundedness for the solutions.

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1 Introduction

Recently there has been a lot of interest in studying the boundedness character and the periodic nature of nonlinear difference equations. For some results in this area, see for example [1-11]. Difference equations have been studied in various branches of mathematics for a long time. First results in qualitative theory of such systems were obtained by Poincaré and Perron in the end of nineteenth and the beginning of twentieth centuries. The systematic description of the theory of difference equations one can find in books [1,6,10].

Cinar [5] gave that the solution of the difference equation

\[ x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}. \]

Karatas et al. [8] gave that the solution of the difference equation

\[ x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}. \]
Aloqeili [2] has obtained the solutions of the difference equation

\[ x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}. \]

In this paper we will give the solutions of the rational difference equation

\[ x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}, \quad (1) \]

\( n = 0, 1, \ldots \) where initial values \( x_0, x_{-1}, \) and \( x_{-2} \) are nonnegative real numbers with \( b x_0 \neq -a \) and \( x_{-1} \neq 0 \). Moreover we investigate some properties for this difference equation such as the local stability and the boundedness for the solutions.

2 Solvability of the difference eq(1) when \( a = 1, \ b = 1 \):

The following theorem give the solution of the difference equation

\[ x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(1 + x_n x_{n-2})} \quad (\ast) \]

**Theorem 1** Suppose that \( \{x_n\} \) be a solution of equation (\ast) where the initial values \( x_0, x_{-1}, \) and \( x_{-2} \) are nonnegative real numbers. Then the solutions of equation (\ast) have the form

\[ x_{4n-3} = \frac{x_0 \ x_{-2}}{x_{-1}(1 - x_0 \ x_{-2})} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 5) \ x_0 \ x_{-2})}{(1 + (4i - 3) \ x_0 \ x_{-2})} \right] \quad (2) \]

\[ x_{4n-2} = x_{-2} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 4) \ x_0 \ x_{-2})}{(1 + (4i - 2) \ x_0 \ x_{-2})} \right] \quad (3) \]

\[ x_{4n-1} = x_{-1} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 3) \ x_0 \ x_{-2})}{(1 + (4i - 1) \ x_0 \ x_{-2})} \right] \quad (4) \]

\[ x_{4n} = x_0 \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 2) \ x_0 \ x_{-2})}{(1 + (4i) \ x_0 \ x_{-2})} \right] \quad (5) \]
where \( n = 1, 2, \ldots \).

**Proof.**

For \( n = 1 \) the result holds. Now suppose that \( n > 0 \) and that our assumption holds for \( n - 1 \). That is,

\[
x_{4n-7} = \frac{x_0 \ x_{-2}}{x_{-1} \ (1 - x_0 \ x_{-2})} \prod_{i=1}^{n-1} \left[ \frac{(1 + (4i - 5) \ x_0 \ x_{-2})}{(1 + (4i - 3) \ x_0 \ x_{-2})} \right]
\]

(6)

\[
x_{4n-6} = x_{-2} \prod_{i=1}^{n-1} \left[ \frac{(1 + (4i - 4) \ x_0 \ x_{-2})}{(1 + (4i - 2) \ x_0 \ x_{-2})} \right]
\]

(7)

\[
x_{4n-5} = x_{-1} \prod_{i=1}^{n-1} \left[ \frac{(1 + (4i - 3) \ x_0 \ x_{-2})}{(1 + (4i - 1) \ x_0 \ x_{-2})} \right]
\]

(8)

\[
x_{4n-4} = x_0 \prod_{i=1}^{n-1} \left[ \frac{(1 + (4i - 2) \ x_0 \ x_{-2})}{(1 + (4i) \ x_0 \ x_{-2})} \right]
\]

(9)

Now we will try to prove that the eqs(2-5) hold at \( n \).

Using Eq(7), Eq(8) and Eq(9) we have

\[
x_{4n-3} = \frac{x_{4n-4} \ x_{4n-6}}{x_{4n-5} \ (1 + x_{4n-4} \ x_{4n-6})}
\]

\[
= \frac{x_0 \ \prod_{i=1}^{n-1} \left[ \frac{(1+(4i-2)x_0 \ x_{-2})}{(1+(4i)x_0 \ x_{-2})} \right]}{x_{-2} \ \prod_{i=1}^{n-1} \left[ \frac{(1+(4i-4)x_0 \ x_{-2})}{(1+(4i-2)x_0 \ x_{-2})} \right]}
\]

\[
\times \ \frac{1}{\left(1 + \left( x_0 \ \prod_{i=1}^{n-1} \left[ \frac{(1+(4i-2)x_0 \ x_{-2})}{(1+(4i)x_0 \ x_{-2})} \right] \right) \ \left( x_{-2} \ \prod_{i=1}^{n-1} \left[ \frac{(1+(4i-4)x_0 \ x_{-2})}{(1+(4i-2)x_0 \ x_{-2})} \right] \right) \right)}
\]

\[
= \frac{x_0 \ x_{-2}/(1 + (4n - 4) \ x_0 \ x_{-2})}{x_{-1} \ \prod_{i=1}^{n-1} \left[ \frac{(1+(4i-3)x_0 \ x_{-2})}{(1+(4i-1)x_0 \ x_{-2})} \right]}
\]
\[
\frac{x_0 x_{-2}}{x_{-1}} \times \prod_{i=1}^{n-1} \left[ \frac{1 + (4i - 1)x_0 x_{-2}}{1 + (4i - 3)x_0 x_{-2}} \right] = \frac{x_0 x_{-2}}{x_{-1}} \times \prod_{i=1}^{n-1} \left[ \frac{1 + (4i - 1)x_0 x_{-2}}{1 + (4i - 3)x_0 x_{-2}} \right] = \frac{x_0 x_{-2}}{x_{-1}} \times \prod_{i=1}^{n-1} \left[ \frac{1 + (4i - 5)x_0 x_{-2}}{1 + (4i - 3)x_0 x_{-2}} \right]
\]

Using Eq(2), Eq(8) and Eq(9) we have

\[
x_{4n-2} = \frac{x_{4n-3} x_{4n-5}}{1 + x_{4n-3} x_{4n-5}}
\]

\[
= \left\{ \frac{x_0 x_{-2}}{x_{-1} x_{-2}} \prod_{i=1}^{n-1} \left[ \frac{1 + (4i - 5)x_0 x_{-2}}{1 + (4i - 3)x_0 x_{-2}} \right] \right\} \left\{ \frac{x_{-1} x_{-2}}{1 + x_{-1} x_{-2}} \prod_{i=1}^{n-1} \left[ \frac{1 + (4i - 3)x_0 x_{-2}}{1 + (4i - 1)x_0 x_{-2}} \right] \right\}
\]

\[
\times \left[ 1 + \left\{ \frac{x_0 x_{-2}}{x_{-1} (1 - x_0 x_{-2})} \prod_{i=1}^{n-1} \left[ \frac{1 + (4i - 5)x_0 x_{-2}}{1 + (4i - 3)x_0 x_{-2}} \right] \right\} \left\{ \frac{x_{-1} x_{-2}}{(1 - x_0 x_{-2}) (1 + (4i - 1)x_0 x_{-2})} \prod_{i=1}^{n-1} \left[ \frac{1 + (4i - 3)x_0 x_{-2}}{1 + (4i - 1)x_0 x_{-2}} \right] \right\} \right]
\]

\[
= \left\{ \frac{x_0 x_{-2}}{x_{-1} (1 - x_0 x_{-2})} \prod_{i=1}^{n-1} \left[ \frac{1 + (4i - 2)x_0 x_{-2}}{1 + (4i - 4)x_0 x_{-2}} \right] \right\} \left\{ \frac{x_{-1} x_{-2}}{(1 - x_0 x_{-2}) (1 + (4i - 3)x_0 x_{-2})} \prod_{i=1}^{n-1} \left[ \frac{1 + (4i - 3)x_0 x_{-2}}{1 + (4i - 1)x_0 x_{-2}} \right] \right\}
\]
\[
\begin{align*}
\text{Rational difference equation} & = \left\{ \frac{x_{-2}}{1 + \left\{ \frac{x_{0}x_{-2}}{(1 + (4n-3)x_{0}x_{-2})} \right\}} \right\} \prod_{i=1}^{n-1} \left\{ \frac{(1 + (4i)x_{0}x_{-2})}{(1 + (4i - 2)x_{0}x_{-2})} \right\} \\
& = \frac{x_{-2}}{1 + (4n - 3)x_{0}x_{-2} + x_{0}x_{-2}} \prod_{i=1}^{n-1} \left\{ \frac{(1 + (4i)x_{0}x_{-2})}{(1 + (4i - 2)x_{0}x_{-2})} \right\} \\
& = x_{-2} \prod_{i=1}^{n} \left\{ \frac{(1 + (4i - 4)x_{0}x_{-2})}{(1 + (4i - 2)x_{0}x_{-2})} \right\} \\
\text{Using Eq}(2), \text{Eq}(3) \text{ and Eq}(9) \text{ we have} \\
x_{4n-1} & = \frac{x_{4n-2}x_{4n-4}}{x_{4n-3}(1 + x_{4n-2}x_{4n-4})} \\
& = \left\{ x_{-2} \prod_{i=1}^{n} \left\{ \frac{(1 + (4i-4)x_{0}x_{-2})}{(1 + (4i-2)x_{0}x_{-2})} \right\} \right\} \left\{ x_{0} \prod_{i=1}^{n-1} \left\{ \frac{(1 + (4i-2)x_{0}x_{-2})}{(1 + (4i)x_{0}x_{-2})} \right\} \right\} \\
& \times \left\{ \frac{1}{1 + \left\{ x_{-2} \prod_{i=1}^{n} \left\{ \frac{(1 + (4i-4)x_{0}x_{-2})}{(1 + (4i-2)x_{0}x_{-2})} \right\} \right\} \left\{ x_{0} \prod_{i=1}^{n-1} \left\{ \frac{(1 + (4i-2)x_{0}x_{-2})}{(1 + (4i)x_{0}x_{-2})} \right\} \right\} \right\} \\
& = \left\{ \prod_{i=1}^{n} \left\{ \frac{(1 + (4i-5)x_{0}x_{-2})}{(1 + (4i-3)x_{0}x_{-2})} \right\} \right\} \left\{ 1 + x_{0}x_{-2} \left\{ \frac{(1 + (4n-4)x_{0}x_{-2})}{(1 + (4n-2)x_{0}x_{-2})} \right\} \right\} \\
& \times \left\{ \prod_{i=1}^{n-1} \left\{ \frac{(1 + (4i-4)x_{0}x_{-2})}{(1 + (4i)x_{0}x_{-2})} \right\} \right\} \\
& = \left\{ \prod_{i=1}^{n} \left\{ \frac{(1 + (4i-5)x_{0}x_{-2})}{(1 + (4i-3)x_{0}x_{-2})} \right\} \right\} \left\{ 1 + x_{0}x_{-2} \left\{ \frac{(1 + (4n-4)x_{0}x_{-2})}{(1 + (4n-2)x_{0}x_{-2})} \right\} \right\} \\
& \times \left\{ \prod_{i=1}^{n-1} \left\{ \frac{1}{1 + (4n-4)x_{0}x_{-2}} \right\} \right\} \\
& = \left\{ \prod_{i=1}^{n} \left\{ \frac{1}{1 + (4n-2)x_{0}x_{-2}} \right\} \right\} \left\{ 1 + x_{0}x_{-2} \left\{ \frac{1}{1 + (4n-2)x_{0}x_{-2}} \right\} \right\} \\
& = \left\{ \prod_{i=1}^{n} \left\{ \frac{1}{1 + (4i-3)x_{0}x_{-2}} \right\} \right\} \left\{ 1 + x_{0}x_{-2} \left\{ \frac{1}{1 + (4i-3)x_{0}x_{-2}} \right\} \right\} \\
& = \left\{ \prod_{i=1}^{n} \left\{ \frac{1}{1 + (4i-3)x_{0}x_{-2}} \right\} \right\} \prod_{i=1}^{n} \left\{ \frac{1 + (4i - 3)x_{0}x_{-2}}{(1 + (4i - 3)x_{0}x_{-2})} \right\} \\
& = \left\{ \prod_{i=1}^{n} \left\{ \frac{1 + (4i - 3)x_{0}x_{-2}}{(1 + (4i - 3)x_{0}x_{-2})} \right\} \right\} \\
\end{align*}
\]
\[
\begin{align*}
&= \frac{x_{n-1} (1 - x_0 x_{-2})}{[(1 + (4n - 1)x_0 x_{-2})] \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 3)x_0 x_{-2})}{(1 + (4i - 5)x_0 x_{-2})} \right]} \\
&= \frac{x_{n-1} (1 - x_0 x_{-2})}{(1 - x_0 x_{-2})} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 3)x_0 x_{-2})}{(1 + (4i - 1)x_0 x_{-2})} \right] \\
&= x_{n-1} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 3)x_0 x_{-2})}{(1 + (4i - 1)x_0 x_{-2})} \right] \\
\end{align*}
\]

Using Eq(2), Eq(3) and Eq(4) we have

\[
x_{4n} = \frac{x_{4n-1} x_{4n-3}}{x_{4n-2} (1 + x_{4n-1} x_{4n-3})}
\]

\[
= \frac{x_{0}}{(1 - x_0 x_{-2})} \left\{ \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 5)x_0 x_{-2})}{(1 + (4i - 1)x_0 x_{-2})} \right] \right\} \\
\]

\[
= \frac{x_{0}}{(1 + (4n - 1)x_0 x_{-2})} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 2)x_0 x_{-2})}{(1 + (4i - 4)x_0 x_{-2})} \right] \left(1 + \frac{x_0 x_{-2}}{(1 + (4n - 1)x_0 x_{-2})} \right) \\
= \frac{x_{0}}{((1 + (4n - 1)x_0 x_{-2}) + x_0 x_{-2})} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 2)x_0 x_{-2})}{(1 + (4i - 4)x_0 x_{-2})} \right] \\
= \frac{x_{0}}{(1 + (4n)x_0 x_{-2})} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 2)x_0 x_{-2})}{(1 + (4i - 4)x_0 x_{-2})} \right] \\
= x_{0} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 2)x_0 x_{-2})}{(1 + (4i)x_0 x_{-2})} \right]
\]

Hence the proof is complete.
Lemma 2 Let $\rho = x_0 \ x_{-2}$. We have the following relations between the solutions in eqs(2-5)

\begin{enumerate}
\item $x_{4n-3} \ x_{4n-1} = \frac{\rho}{1 + (4n - 1) \rho}$
\item $x_{4n} \ x_{4n-2} = \frac{1 + 4n \rho}{1 + 4n \rho}$
\item $\frac{1}{x_{4n}} \frac{1}{x_{4n-2}} - \frac{1}{x_{4n-3}} \frac{1}{x_{4n-1}} = 1$
\end{enumerate}

Proof.

i) From Eq(2) and Eq(4) we have

$$x_{4n-3} \ x_{4n-1} = \left\{ \frac{x_0 \ x_{-2}}{x_{-1} (1 - x_0 \ x_{-2})} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 5) x_0 \ x_{-2})}{(1 + (4i - 3) x_0 \ x_{-2})} \right] \right\}$$

$$\times \left\{ x_{-1} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 3) x_0 \ x_{-2})}{(1 + (4i - 1) x_0 \ x_{-2})} \right] \right\}$$

$$= \left\{ \frac{x_0 \ x_{-2}}{1 - x_0 \ x_{-2}} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 5) x_0 \ x_{-2})}{(1 + (4i - 3) x_0 \ x_{-2})} \right] \right\}$$

$$= \frac{\rho}{1 - \rho} \frac{(1 - \rho)(1 + 3\rho) \ldots (1 + (4n - 5) \rho)}{(1 - \rho)(1 + 3\rho)(1 + 7\rho) \ldots (1 + (4n - 1) \rho)}$$

$$= \frac{\rho}{1 - \rho} \frac{(1 - \rho)}{(1 + (4n - 1) \rho)} = \frac{\rho}{1 + (4n - 1) \rho}$$

ii) From Eq(3) and Eq(5) we have

$$x_{4n} \ x_{4n-2} =$$

$$= \left\{ x_0 \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 2) x_0 \ x_{-2})}{(1 + (4i) x_0 \ x_{-2})} \right] \right\} \left\{ x_{-2} \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 4) x_0 \ x_{-2})}{(1 + (4i - 2) x_0 \ x_{-2})} \right] \right\}$$

$$= x_0 x_{-2} \left\{ \prod_{i=1}^{n} \left[ \frac{(1 + (4i - 4) x_0 \ x_{-2})}{(1 + (4i) x_0 \ x_{-2})} \right] \right\}$$
\[ = \rho \frac{(1) (1 + 4\rho) \ldots \ldots (1 + (4n - 4) \rho)}{(1 + 4\rho) (1 + 8\rho) \ldots \ldots (1 + (4n) \rho)} = \frac{\rho}{1 + (4n) \rho} \]

iii) By easy calculations from i) and ii).

**Remark 1** We note that
\[ \frac{x_{4n-3} x_{4n-2}}{x_{4n-3} x_{4n-1}} = \frac{1 + (4n-1) \rho}{1 + 4n \rho}. \]

**Remark 2** We note that \(|x_{4n-3} x_{4n-1}| \to 0 \text{ as } n \to \infty .\)

**Remark 3** We note that \(|x_{4n} x_{4n-2}| \to 0 \text{ as } n \to \infty .\)

### 3 Solvability of the difference eq(1) when \(a = 1, \ b = -1 :\)

The following theorem give the solution of the difference equation

\[ x_{n+1} = \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-2}}{1 - x_n x_{n-2}}. \]

**Theorem 3** Suppose that \(\{x_n\}\) be a solution of Equation (1) and \(x_0 = h, \ x_{-1} = k \text{ and } x_{-2} = r.\) Let \(\rho = x_0 x_{-2}.\) Then the solutions of equation (1) have the form

\[ x_{4n-3} = h \frac{r}{k (1 + \rho)} \prod_{i=1}^{n} \frac{(1 - (4i - 5) \rho)}{(1 - (4i - 3) \rho)} \]

\[ x_{4n-2} = r \prod_{i=1}^{n} \frac{(1 - (4i - 4) \rho)}{(1 - (4i - 2) \rho)} \]

\[ x_{4n-1} = k \prod_{i=1}^{n} \frac{(1 - (4i - 3) \rho)}{(1 - (4i - 1) \rho)} \]

\[ x_{4n} = h \prod_{i=1}^{n} \frac{(1 - (4i - 2) \rho)}{(1 - (4i) \rho)} \]

where \(n = 1, 2, \ldots .\)

**Proof.** By using the mathematical induction as in theorem 1.
4 Solvability of the difference eq(1) when $a \neq 1$

The following theorem give the solution of the difference equation (1) where $a \neq 1$ and for any value of $b$.

**Theorem 4** Suppose that $\{x_n\}$ be a solution of Equation (1) and $x_0 = h$, $x_{-1} = k$, $x_{-2} = r$ and $\rho = x_0 x_{-2}$. Then the solutions of equation (1) have the form

$$x_1 = \frac{\rho}{k (a + b \rho)} , \quad x_2 = \frac{r}{a^2 + 2 a b \rho}$$

$$x_3 = \frac{k (a + b \rho)}{a^3 + (2a^2 + 1) b \rho} , \quad x_4 = \frac{h (a^2 + 2 a b \rho)}{a^4 + (2a^3 + a + 1) b \rho}$$

$$x_{4n-3} = \frac{\rho}{k (a + b \rho)} \prod_{i=2}^{n} \left[ \frac{a^{4i-5} + b \rho \left\{ 2a^{4i-6} + \frac{1-a^{4i-7}}{1-a} \right\}}{a^{4i-3} + b \rho \left\{ 2a^{4i-4} + \frac{1-a^{4i-5}}{1-a} \right\}} \right]$$

$$x_{4n-2} = r \frac{\prod_{i=1}^{n-1} a^{4i} + b \rho \left\{ 2a^{4i-1} + \frac{1-a^{4i-2}}{1-a} \right\}}{\prod_{i=1}^{n} a^{4i-2} + b \rho \left\{ 2a^{4i-3} + \frac{1-a^{4i-4}}{1-a} \right\}}$$

$$x_{4n-1} = k \frac{\prod_{i=2}^{n} a^{4i-3} + b \rho \left\{ 2a^{4i-4} + \frac{1-a^{4i-5}}{1-a} \right\}}{\prod_{i=2}^{n+1} a^{4i-5} + b \rho \left\{ 2a^{4i-6} + \frac{1-a^{4i-7}}{1-a} \right\}}$$

$$x_{4n} = h \frac{\prod_{i=1}^{n} a^{4i-2} + b \rho \left\{ 2a^{4i-3} + \frac{1-a^{4i-4}}{1-a} \right\}}{a^{4i} + b \rho \left\{ 2a^{4i-1} + \frac{1-a^{4i-2}}{1-a} \right\}}$$

where $n = 2, 3, \ldots$ .

**Proof.** By using the mathematical induction as in theorem 1.

**Lemma 5** The solutions $\{x_n\}$ of the difference eqs (1) and (*) have no prime period two solution
Now we recall some notations and results which will be useful in our study.

**Definition 1** The difference equation

\[ x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \]  

(10)

is said to be persistence if there exist numbers \( m \) and \( M \) with \( 0 < m \leq M < \infty \) such that for any initial conditions \( x_{-k}, x_{-k+1}, \ldots, x_1, x_0 \in (0, \infty) \) there exists a positive integer \( N \) which depends on the initial conditions such that

\[ m \leq x_n \leq M \quad \text{for all} \quad n \geq N. \]

**Definition 2** (i) The equilibrium point \( \bar{x} \) of Eq.(10) is locally stable if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x_{-k}, x_{-k+1}, \ldots, x_1, x_0 \in I \) with

\[ |x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \delta, \]

we have

\[ |x_n - \bar{x}| < \epsilon \quad \text{for all} \quad n \geq -k. \]

(ii) The equilibrium point \( \bar{x} \) of Eq.(10) is locally asymptotically stable if \( \bar{x} \) is locally stable solution of Eq.(10) and there exists \( \gamma > 0 \), such that for all \( x_{-k}, x_{-k+1}, \ldots, x_1, x_0 \in I \) with

\[ |x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \ldots + |x_0 - \bar{x}| < \gamma, \]

we have

\[ \lim_{n \to \infty} x_n = \bar{x}. \]

The linearized equation of Eq.(10) about the equilibrium \( \bar{x} \) is the linear difference equation

\[ y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\bar{x}, \bar{x}, \ldots, \bar{x})}{\partial x_{n-i}} y_{n-i}. \]

**Theorem A** [9] Assume that \( p, q \in R \). Then

\[ |p| + |q| < 1 \]

is a sufficient condition for the asymptotic stability of the difference equation

\[ x_{n+1} - px_n - qx_{n-1} = 0, \quad n = 0, 1, \ldots. \]
5 Local Stability of the Equilibrium Points

In this section we study the local stability of the solutions of Eq.(1).

We first give the equilibrium points of Eq(1).

**Lemma 6** The equilibrium points of the difference eq (1) are $0$ and $\pm \sqrt{\frac{a-1}{b}}$

**Proof.**

$$\bar{x} = \frac{x^2}{x(a + bx^2)}$$

$$x^2(a + bx^2) = x^2$$

$$x^2(-1 + a + bx^2) = 0$$

thus the equilibrium points of the difference eq (1) are $0$ and $\pm \sqrt{\frac{a-1}{b}}$.

**Remark 4** When $a = 1$, then the only equilibrium point of the difference eq (*) is $0$.

**Theorem 7** The equilibrium points $\bar{x} = \pm \sqrt{\frac{a-1}{b}}$ are locally asymptotically stable if $a > 1$ and in this case $|a - 2a^2| < 4a^2 - 6a + 1$.

**Proof.**

We will prove the theorem at the equilibrium point $\bar{x} = +\sqrt{\frac{a-1}{b}}$ and the proof at the equilibrium point $\bar{x} = -\sqrt{\frac{a-1}{b}}$ by the same way.

let $f : (0, \infty)^3 \rightarrow (0, \infty)$ be a continuous function defined by

$$f(u,v,w) = \frac{uw}{v(a + bw)}$$

Therefore it follows that

$$\frac{\partial f(u,v)}{\partial u} = \frac{v(a + bw)w - uw(vbw)}{v^2(a + bw)^2} = \frac{aw}{v(a + bw)^2}$$

$$\frac{\partial f(u,v)}{\partial v} = \frac{-uw}{v^2(a + bw)}$$
\[
\frac{\partial f(u, v)}{\partial w} = \frac{au}{v (a + bw)^2}
\]

At the equilibrium point \( x = \sqrt{\frac{a-1}{b}} \) we have

\[
\frac{\partial f(x, x, x)}{\partial u} = \frac{a}{(a + b(\frac{a-1}{b}))^2} = \frac{a}{(2a - 1)^2} = p_1
\]

\[
\frac{\partial f(x, x, x)}{\partial v} = \frac{-a}{(2a - 1)} = p_2
\]

\[
\frac{\partial f(x, x, x)}{\partial w} = \frac{a}{(2a - 1)^2} = p_3
\]

Then the linearized equation of Eq.(1) about \( x = \sqrt{\frac{a-1}{b}} \) is

\[
y_{n+1} - p_1y_n - p_2y_{n-1} - p_3y_{n-2} = 0.
\]

i. e.

\[
y_{n+1} - \frac{a}{(2a - 1)^2}y_n + \frac{a}{(2a - 1)}y_{n-1} - \frac{a}{(2a - 1)^2}y_{n-2} = 0
\]

Whose characteristic equation is

\[
\lambda^3 - \frac{a}{(2a - 1)^2}\lambda^2 + \frac{a}{(2a - 1)}\lambda - \frac{a}{(2a - 1)^2} = 0
\]

By the generalization of theorem A we have

\[
|p_1| + |p_2| + |p_3| < 1
\]

\[
\left| \frac{a}{(2a - 1)^2} \right| + \left| \frac{-a}{(2a - 1)} \right| + \left| \frac{a}{(2a - 1)^2} \right| < 1
\]

\[
|a| + |a (2a - 1)| + |a| < (2a - 1)^2
\]

\[
|a - 2a^2| < 4a^2 - 6a + 1
\]
6 Boundedness of solutions

Here we study the boundedness of Eq. (*) .

Theorem 8 Every solution of Eq. (*) is bounded from above .

Proof: Let \( \{x_n\}_{n=-k}^{\infty} \) be a solution of Eq. (*) . It follows from Eq. (*) that

\[
x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(1 + x_n x_{n-2})} = \frac{x_n x_{n-2}}{x_{n-1} + x_{n-1} x_n x_{n-2}} \leq \frac{1}{x_{n-1}}
\]

Then

\[
x_n \leq \frac{1}{x_{n-2}} \quad \text{for all} \quad n \geq 0.
\]

This means that every solution of eq (*) is bounded from above by \( M = \max(\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}) \).

References

\[ x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}, \] Appl. Math. Comp.(in press), 2003.
\[ x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \] Appl. Math. Comp., 150, (2004), 21-24.
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