The Categories of Textures and Topological Spaces

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Abstract

In this study, mainly, we introduce the category $\text{Tex}$ of textures and texturally continuous functions. Relations with some other categories, particularly $\text{dftTex}$, $\text{ftTex}$, are pointed out. It is shown that the categories $\text{CTop}$ of C-spaces and a full subcategory of $\text{Sober}$ of sober spaces are, respectively, isomorphic to $\text{Tex}$ and $\text{STex}$ of simple textures. Some properties discussed including the existence of product, coproduct, equalizer coequalizer.

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1 Introduction

The theory of texture spaces was introduced by L. M. Brown in 1992 under the name “fuzzy structure”, and results on this topic appear in several papers including [3-7,11,12].

Texture space: [4] Let $S$ be a set. A texturing is a point-separating, complete, completely distributive lattice with respect to inclusion, which contains $S$ and $\emptyset$, and for which arbitrary meets coincide with intersections, and finite joins with unions. If $S$ is a texturing of $S$ the pair $(S, S)$ is called a texture space or simplify a texture for short.

For $s \in S$ the sets $P_s = \bigcap \{A \in S \mid s \in A\}$ and $Q_s = \bigvee \{P_u \mid u \in S, \; s \notin P_u\}$ are called respectively, the p-sets and q-sets of $(S, S)$.

In a texture, arbitrary joins need not coincide with unions, and clearly this will be so if and only if $P_s \notin Q_s$ for all $s \in S$. In this case $(S, S)$ is said to be plain.
On the other hand, $M \in S$ is called a molecule if $M \neq \emptyset$ and $M \subseteq A \cup B$, $A, B \in S$ implies $M \subseteq A$ or $M \subseteq B$. The texture $(S, S)$ is called simple if the sets $P_s$, $s \in S$ are the only molecules in $S$.

**Example 1.1.** 1. For any set $X$, $(X, \mathcal{P}(X))$ is the discrete texture representing the usual set structure of $X$. Clearly, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$, so $(X, \mathcal{P}(X))$ is both plain and simple.

2. Let $(\mathbb{L}, \iota)$ be a Hutton algebra, that is a complete, completely distributive lattice $\mathbb{L}$ equipped with an order-reversing involution $\iota$. If we denote by $M_\mathbb{L}$ the set of molecules in $\mathbb{L}$, set $\hat{a} = \{m \in M_\mathbb{L} \mid m \leq a\}$ for $a \in \mathbb{L}$, and $\mathbb{M}_\mathbb{L} = \{\hat{a} \mid a \in \mathbb{L}\}$, then $(M_\mathbb{L}, \mathbb{M}_\mathbb{L})$ is a simple texture [4].

3. Let $L = (0, 1]$, $\mathcal{L} = \{(0, r) \mid r \in [0, 1]\}$. Clearly $(L, \mathcal{L})$ is the Hutton texture of $(\mathbb{I}, \iota)$, where $\mathbb{I} = [0, 1]$ with its usual order and $r' = 1 - r$ for $r \in \mathbb{I}$. Here $P_r = Q_r = (0, r]$ for all $r \in L$. Although $(L, \mathcal{L})$ is simple it is clearly not plain.

4. Let $B = \{a, b, c\}$, $\mathcal{B} = \{\sigma, \{a\}, \{a, b\}, B\}$. Then $(B, \mathcal{B})$ is a texture. Clearly $P_a = \{a\}$, $P_b = \{a, b\}$, $P_c = B$ and $Q_a = \sigma$, $Q_b = \{a\}$, $Q_c = \{a, b\}$, so $(B, \mathcal{B})$ is both plain and simple.

**Texturally continuity:** [10] Let $(S, S)$ and $(T, \mathcal{T})$ be textures. A function $f : S \to T$ is called texturally continuous from $(S, S)$ to $(T, \mathcal{T})$ if $f^{-1}(V) \in S$ for all $V \in \mathcal{T}$ (it was called affine map in [12]).

Where there can be no confusion we will write continuous function instead of texturally continuous function.

**Natural subtexture:** [10] Let $(S, S)$ be a texture and $U$ be a subset of $S$. A texture $(U, \mathcal{U})$ is called a natural subtexture of $(S, S)$ iff the inclusion function $i : U \hookrightarrow S$ is a continuous function from $(U, \mathcal{U})$ to $(S, S)$ such that for all texture $(T, \mathcal{T})$ and function $f : T \to S$, if $i \circ f : T \to S$ is a continuous function, then so is $f$.

The texturing $\mathcal{U}$ on $U$ is denoted by $S_U$.

**Quotient texture:** [5] Let $(S, S)$ be a texture and $\sim$ an equivalence relation on $S$ and $q : S \to U = S/ \sim$ the canonical function sending each point of $S$ to its equivalence class w.r.t., $\sim$. If $\mathcal{U} = \{A \subseteq U \mid q^{-1}(A) \in S\}$ is a texturing of $U$, $(U, \mathcal{U})$ is called a quotient texture of $(S, S)$.

From the definition it is clear that the quotient function $q$ is continuous.

**Product texture:** [4] If $(S_j, S_j)$, $j \in J$, are textures, $S = \prod_{j \in J} S_j$ and $A_k \in S_k$ for some $k \in J$ we write

$$E(k, A_k) = \prod_{j \in J} Y_j$$

where $Y_j = \begin{cases} A_j, & \text{if } j = k \\ S_j, & \text{otherwise.} \end{cases}$
Then the product texturing $S = \bigotimes_{j \in J} S_j$ on $S$ consists of arbitrary intersections of elements of the set $E = \left\{ \bigcup_{j \in J_1} E(j, A_j) \mid J_1 \subseteq J, A_j \in S_j \text{ for } j \in J_1 \right\}$.

**Sum texture:** [6] If for the textures $(S_j, S_j)$, $j \in J$, the sets $S_j$ are not necessarily pairwise disjoint, we shall mean by the disjoint sum of the textures $(S_j, S_j)$ the sum of the disjoint textures $(S_j \times \{j\}, \{A \times \{j\} \mid A \in S_j\})$. Hence the disjoint sum is $(S, S)$ where $S = \bigcup_{j \in J}(S_j \times \{j\})$ and $S = \{A \subseteq S \mid A \cap (S_j \times \{j\}) = A_j \times \{j\} \implies A_j \in S_j, \forall j \in J\}$.

**Difunction:** [5] A difunction from $(S, S)$ to $(T, T)$ is a direlation [5] $(f, F)$ from $(S, S)$ to $(T, T)$ satisfying the conditions:

(DF1) For $s, s' \in S$, $P_s \notin Q_{s'} \implies \exists t \in T$ with $f \notin \overline{Q}_{(s, t)}$ and $\overline{P}_{(s', t)} \notin F$.

(DF2) For $t, t' \in T$ and $s \in S$, $f \notin \overline{Q}_{(s, t)}$ and $\overline{P}_{(s, t')} \notin F \implies P_t \notin Q_t$.

The category $\text{dfTex}$ of textures and difunctions was introduced in [5]. If we take as objects simple, plain, plain-simple textures we obtain, respectively, the full subcategory $\text{dfSTex}$, $\text{dfPTex}$, $\text{dfPSTex}$.

Let $f : X \to Y$ be any ordinary point function and $f' = f^c = (X \times Y) \setminus f$. Then the mapping $\mathcal{T} : \text{Set} \to \text{dfPSTex}$ where $\mathcal{T}(S) = (S, \mathcal{P}(S))$ and $\mathcal{T}(f) = (f, f')$ is an embedding functor [5].

**Proposition 1.2.** [5] Let $(S, S)$ be a texture and $(T, \mathcal{T})$ a simple texture. Suppose that the point function $\varphi$ on $S$ to $T$ satisfies the conditions

(a) $P_s \notin Q_{s'} \implies P_{\varphi(s)} \notin Q_{\varphi(s')}$ for all $s, s' \in S$.

(b) $P_{\varphi(s)} \notin B, B \in \mathcal{T} \implies \exists s' \in S$ with $P_s \notin Q_{s'}$ for which $P_{\varphi(s')} \notin B$.

Then the equalities

$$f = f_\varphi = \bigcup\{\overline{P}_{(s, t)} \mid \exists u \in S \text{ satisfying } P_s \notin Q_u \text{ and } P_{\varphi(u)} \notin Q_t\},$$

$$F = F_\varphi = \bigcap\{\overline{P}_{(s, t)} \mid \exists u \in S \text{ satisfying } P_u \notin Q_s \text{ and } P_t \notin Q_{\varphi(u)}\},$$

define a difunction $(f, F)$ on $(S, S)$ to $(T, \mathcal{T})$. Moreover, for $B \in \mathcal{T}$, $F^{-}B = \varphi^{-}B = f^{-}B$, where $\varphi^{-}B = \bigcup\{P_s \mid P_{\varphi(u)} \subseteq B \forall u \in S \text{ with } P_s \notin Q_u\}$.

Conversely, if $\varphi : S \to T$ is any function satisfying (a) and (b) then there exists a unique difunction $(f_\alpha, F_\alpha) : (S, S) \to (T, \mathcal{T})$ satisfying $\varphi = \varphi(f_\alpha, F_\alpha)$.

If we consider plain textures we obtain the same class of point functions.
**Proposition 1.3.** [5] The function \( \psi : S \to T \) corresponding as above to the difunction \((f, F) : (S, S) \to (T, T)\), with \((S, S)\) plain, has the properties (a) and (b).

Conversely, if \( \psi : S \to T \) is any function satisfying (a) and (b) then there exists a unique difunction \((f, F) : (S, S) \to (T, T)\) satisfying \( \psi = \psi_{(f, F)} \).

The category of textures and point functions between the base sets satisfying the conditions (a)-(b) denoted by \( f_{\text{Tex}} \) in [5]. Likewise, if we take as objects simple, plain, plain-simple textures we obtain, respectively, the full subcategory \( f_{\text{STex}}, f_{\text{PTex}}, f_{\text{PSTex}} \).

From the Propositions 1.2 and 1.3, the categories \( f_{\text{STex}}, f_{\text{PTex}}, f_{\text{PSTex}} \) are, respectively, isomorphic to \( df_{\text{STex}}, df_{\text{PTex}}, df_{\text{PSTex}} \) ([5], Theorem 3.10).

We note that our general references for concepts from category theory are [1,14]. If \( A \) is a category, \( \text{Ob} A \) will denote the class of objects and \( \text{Mor} A \) the class of morphisms of \( A \). We will sometimes use the notation \( A(A_1, A_2) \) for the set of morphisms in \( A \) from \( A_1 \) to \( A_2 \).

## 2 The category \( \text{Tex} \)

In this section, we introduce the category which, will mainly concern us in this study.

Let \( \emptyset \) be the class of textures. Now we consider the set

\[
\text{hom}((S, S), (T, T)) = \{ f \mid f : S \to T \text{ is a continuous function} \},
\]

for all the pair \((S, S), (T, T)\) \( \in \emptyset \times \emptyset \). On the other hand, the identity function \( id_S : S \to S \) is continuous, for some \((S, S) \in \emptyset \). So, the quadruple \((\emptyset, \text{hom}, id, \circ)\) is a category, where \( \circ \) is usually composition operator. Hence we have:

**Theorem 2.1.** Textures and continuous functions form a category.

This justifies the following definition:

**Definition 2.2.** The category whose objects are textures and whose morphisms are continuous functions will be denoted by \( \text{Tex} \).

If the objects are restricted to be simple, plain, plain-simple we obtain, respectively, the full subcategories \( \text{STex}, \text{PTex}, \text{PSTex} \).
Let $S$ be a set. Now we consider the plain simple texture $(S, \mathcal{P}(S))$ of Example 1.1.(1). Hence $\mathcal{T}(S) = (S, \mathcal{P}(S))$ defines a mapping from the objects of the category $\textbf{Set}$ to those of $\textbf{PSTex}$. If $S, T$ are sets and $f : S \to T$ a point function, then it is also continuous function from $(S, \mathcal{P}(S))$ to $(T, \mathcal{P}(T))$. Hence $\mathcal{T}(f) = f$ gives a mapping from the morphisms of $\textbf{Set}$ to those of $\textbf{PSTex}$. Clearly, the mapping $\mathcal{T} : \textbf{Set} \to \textbf{PSTex}$ is an embedding functor.

Now let the mapping $\mathcal{G} : \textbf{Tex} \to \textbf{Set}$ be defined,

$$\mathcal{G}(S, S) = S, \quad \mathcal{G}(f) = f.$$ 

Clearly, $\mathcal{G}$ is a forgetful functor. So, $\textbf{Tex}$ is a concrete category over $\textbf{Set}$.

**Proposition 2.3.** The functor $\mathcal{G}$ which is defined above is an adjoint.

**Proof.** Let us take $S \in \text{Ob} \textbf{Set}$. Then the pair $(\iota_S, (S, \mathcal{P}(S)))$ is an $\mathcal{G}$-universal arrow with domain $S$. Indeed, the identity function $\iota_S : S \to S$ is clearly an $\mathcal{G}$-structured arrow with domain $S$ because $\mathcal{G}(S, \mathcal{P}(S)) = S$ and $\iota_S$ is a $\textbf{Set}$ morphism. Now we show the universal property. Take $(T, \mathcal{T}) \in \text{Ob} \textbf{Tex}$ and let $f : S \to \mathcal{G}(T, \mathcal{T}) = T$ be a morphism in $\textbf{Set}$. $f$ is continuous function from $(S, \mathcal{P}(S))$ to $(T, \mathcal{P}(T))$. Hence $f \in \text{Mor} \textbf{Tex}$, and this is clearly the unique morphism which makes the following diagram commutative.

$$
\begin{array}{ccc}
S & \xrightarrow{\iota_S} & \mathcal{G}(S, \mathcal{P}(S)) = S \\
\downarrow f & & \downarrow \mathcal{G}(f) \\
\mathcal{G}(T, \mathcal{P}(T)) & & \\
\end{array}
$$

In the category $\textbf{fTex}$, isomorphisms are texturally isomorphisms ([5], Proposition 3.15). If $f : S \to T$ is a morphism in $\textbf{fTex}$, then $f^{-1}(B) \in \mathcal{G}$ for all $B \in \mathcal{T}$ [6, Lemma 3.9]. Hence $\textbf{fTex}$ is an isomorphism-closed subcategory of $\textbf{Tex}$. Likewise, the categories $\textbf{fSTex}$, $\textbf{fPTex}$, $\textbf{fPSTex}$ are, respectively, an isomorphism-closed subcategory of $\textbf{STex}$, $\textbf{PTex}$, $\textbf{PSTex}$.

In the following diagram, $\mathcal{V}$ is isomorphism functor and $\varepsilon$ is inclusion functor.

$$
\begin{array}{ccc}
d\textbf{fSTex} & \xrightarrow{\mathcal{V}} & \textbf{fSTex} \\
\varepsilon & \xrightarrow{} & \textbf{STex} \\
\end{array}
$$

So, $d\textbf{fSTex}$ is a subcategory of $\textbf{STex}$. Likewise, the categories $d\textbf{fPTex}$, $d\textbf{fPSTex}$ are, respectively, subcategory of $\textbf{PTex}$, $\textbf{PSTex}$.

We have:
Theorem 2.4. The functor $\mathcal{G}$ is an adjoint of $\mathcal{E} \circ \mathcal{V} \circ \mathcal{I}$.

Proof. Immediate from [5, Theorem 3.12].

We end this section some results about the morphisms of $\mathbf{Tex}$. The reader is referred to [1,14] for the relevant terms from category theory.

Proposition 2.5. Let $(S, S), (T, T)$ be textures and $f : S \to T$ a morphism in the category $\mathbf{Tex}$.

1. $f$ is injective iff $f$ is monomorphism.
2. If $f$ is surjective then $f$ is epimorphism.
3. If $f$ is a section then it is injective.
4. If $f$ is a retraction then it is surjective.
5. $f$ is isomorphism iff $f$ is bijective and $f^{-1}$ is continuous, i.e., $f$ is texturally isomorphism in the sense of [6].

Proof. (1)(⇒) Suppose that $f \circ \gamma = f \circ \delta$, for $\gamma, \delta \in \mathbf{Tex}((Z, Z), (S, S))$. We must show $\gamma = \delta$. Since $f$ is injective, $(f \circ \gamma)(z) = (f \circ \delta)(z) \implies f(\gamma(z)) = f(\delta(z)) \implies \gamma(z) = \delta(z), \forall z \in Z$. Hence, we obtain $\gamma = \delta$.

(⇐) Suppose that $f(x) = f(y)$, for all $x, y \in S$. Now we consider the functions $\tilde{x}, \tilde{y} : S \to S$, where

$$\tilde{x}(z) = x, \quad \tilde{y}(z) = y, \quad \forall z \in S.$$  

Clearly, $\tilde{x}, \tilde{y}$ are continuous and $f \circ \tilde{x} = f \circ \tilde{y}$. Since $f$ is monomorphism, $\tilde{x} = \tilde{y}$, hence we obtain $x = y$.

(2) Suppose that $\alpha \circ f = \beta \circ f$ for $\alpha, \beta \in \mathbf{Tex}((T, T), (S, S))$. We must show $\alpha = \beta$. Since $f$ surjective, there exists $t \in T$ such that $f(s) = t$, for all $s \in S$. Hence, for $t \in T$, $\alpha(t) = \alpha(f(s)) = \beta(f(s)) = \beta(t)$. So we have $\alpha = \beta$.

(3) Suppose $f$ is a section. Then there exists a morphism $g \in \mathbf{Tex}((T, T), (S, S))$ such that $g \circ f = \iota_S$. For some $x, y \in S$, we obtain $f(x) = f(y) \implies g(f(x)) = g(f(y)) \implies (g \circ f)(x) = (g \circ f)(y) \implies \iota_S(x) = \iota_S(y) \implies x = y$. Hence $f$ is injective.

(4) Suppose $f$ is a retraction. Then there exists a morphism $g \in \mathbf{Tex}((T, T), (S, S))$ such that $f \circ g = \iota_T$. Take $t \in T$. Then $f$ is surjective because $g(t) \in S$ and $f(g(t)) = t = \iota_T(t)$.

(5) It is clear.  \qed
3 Textures and Topologies

In this section, we will give the various categorial relations between textures and topological space.

Now let \((S, S)\) be a texture. Then \(S^c = \{ S \setminus A \mid A \in S \}\) is a topology that is \(S\) is the family of closed sets for a topology \(S^c\) on \(S\). Furthermore, if \(f : (S, S) \to (T, T)\) is a morphism in the category \(\text{Tex}\), then, \(f : (S, S^c) \to (T, T^c)\) is a morphism in the category \(\text{Top}\) of topological spaces and continuous functions. Hence, the category \(\text{Tex}\) is full embeddable into \(\text{Top}\).

Since a texturing separates points the corresponding topology is \(T_0\). Hence there is a natural relation between texture spaces and \(T_0\) topological spaces.

Now we form a full subcategory of \(\text{Top}_0\) of \(T_0\) topological spaces: Let \((S, T)\) be \(T_0\) topological space and \(K = \{ K_i^j \subseteq S \mid i \in I, j \in J \subseteq I, K_i^j \text{ closed} \}\) where

\[
\bigcap_{i \in I} \bigcup_{j \in J_i} K_i^j = \bigcup_{\gamma \in \Pi_i J_i} \bigcap_{i \in I} K_i^\gamma \quad \text{......(*)}
\]

Since \(\bigvee_{j \in J_i} K_i^j = \bigcup_{j \in J_i} K_i^j\) we obtain the below equalities

\[
\bigwedge_{i \in I} \bigvee_{j \in J_i} K_i^j = \bigvee_{\gamma \in \Pi_i J_i} \bigwedge_{i \in I} K_i^\gamma.
\]

Hence the family \(S = \{ K \subseteq S \mid S \setminus K \in \mathcal{I} \}\) is a texturing on \(S\).

In [10], any \(T_0\) topological space which is satisfied (*) is called C-spaces, i.e., \(T_0\) topological space with completely distributive lattice of open (or closed) sets.

We denote by \(\text{CTop}\) the category of C-spaces and continuous functions.

**Theorem 3.1.** The category \(\text{Tex}\) is isomorphic to the category \(\text{CTop}\).

**Proof.** Let \(\mathcal{F} : \text{Tex} \to \text{CTop}\) be defined by

\[
\mathcal{F}(S, S) = (S, S^c) \quad \text{and} \quad \mathcal{F}(f) = f,
\]

where \((S, S) \in \text{Ob}(\text{Tex})\), \(f \in \text{Mor}(\text{Tex})\). Clearly, \(\mathcal{F}\) is a functor. Now we consider the hom-set restriction \(\mathcal{F} : \text{Tex}((S, S), (T, T^c)) \to \text{CTop}((S, S^c), (T, T^{c*}))\). Bearing in mind the fact that \(S = \{ S \setminus U \mid U \in S^c \}\) and \(S^* = \{ S \setminus V \mid V \in S^{c*} \}\), the functor \(\mathcal{F}\) is isomorphism.

Now we will investigate the relations between simple textures and topological spaces.
Let \((S, S)\) be texture and let \(\sim\) be the equivalence relation on \(S\), given by:

\[ x \sim y \iff P_x = P_y. \]

Let \(U := S/\sim\), \(q : S \to U\) canonical function and \(U\) be quotient texturing on \(U\). If we consider the function \(q : (S, S^c) \to (U, U^c)\) between the topological spaces, we have:

1. \(q\) both open and closed function.
2. Let \(A \subseteq S\) be open (closed) set. If \(x \in A\) and \(x \sim y \Rightarrow y \in A\).
3. If \(A \subseteq S\) is open (closed) set then \(q^{-1}(q(A)) = A\).

If we use these properties, then we have:

**Proposition 3.2.** \((S, S)\) simple texture if and only if the quotient texture \((U, U)\) is simple.

**Proof.** (\(\Longrightarrow\)) Suppose \((S, S)\) simple texture. Let \(F \in U\) be a molecule. Since \(q\) is surjective, \(q^{-1}(F)\) is a molecule in \((S, S)\). Then there exists \(s \in S\) such that \(q^{-1}(F) = P_s\). Hence, we have \(F = q(P_s) = P_{q(s)}\).

(\(\Longleftarrow\)) Suppose \((U, U)\) simple texture. Let \(F \in S\) be a molecule. Then \(q(F)\) is a molecule in \((U, U)\). So, there exists \(u \in U\) such that \(q(F) = P_{q(u)}\). Then we obtain \(F = P_u\).

Recall that a space \(Y\) is called irreducible iff it is not the union of the nonempty closed sets. A space \(X\) is called sober ([13], spectral space) iff every irreducible, nonempty, closed subset \(M\) of \(X\) has a unique generic point \(m\), i.e., \(M = \{m\}\) [9]. Clearly, every sober space is a \(T_0\) space.

**Lemma 3.3.** \((S, S)\) is a simple texture if and only if the corresponding topological space \((S, S^c)\) is a sober space.

**Proof.** (\(\Longrightarrow\)) Suppose \((S, S)\) simple texture. Let \(F\) be an irreducible closed set in \((S, S^c)\). We show that \(F\) is a molecule. Let \(F \subseteq K_1 \cup K_2\), for some \(K_1, K_2 \in S\). We note that the sets \(F \cap K_j\) are closed, for \(j = 1, 2\). If \(F \not\subseteq K_1\) and \(F \not\subseteq K_2\) then we can write \(F = (F \cap K_1) \cup (F \cap K_2)\). The last equality is a contradiction because \(F\) is irreducible set. So, \(F\) is a molecule in \((S, S)\). Hence there exists \(s \in S\) such that \(F = P_s\). Since \(P_s = \{s\}\), the topological space \((S, S^c)\) is sober.

(\(\Longleftarrow\)) We suppose \((S, S^c)\) is sober space. Let \(F \in S\) be a molecule. Then \(F\) is a irreducible set in \((S, S^c)\). Hence there exists \(s \in S\) such that \(F = \{s\}\). Since \(P_s = \{s\}\), the proof is complete.
We denote by $\text{CSober}$ the category of sober topological space which are satisfied above (*) equality and continuous functions. Clearly, $\text{CSober}$ is a full subcategory of $\text{Sober}$ of sober topological spaces.

From the same way of the proof of Theorem 3.1. we have:

**Theorem 3.4.** The category $\text{STex}$ is isomorphic to the category $\text{CSober}$.

It will be noted that that requirement that a texturing be completely distributive imposes a condition on the corresponding topological space which is not always satisfied, so the above isomorphisms are proper.

In most of the concrete categories the epimorphisms are precisely the morphisms which are surjective on the underlying sets [8,15]. However, in the category $\text{Top}_0$ of $T_0$ topological spaces, epimorphisms are b-dense mappings [8]. The following theorem and corollary were proved by Baron in [2].

**Theorem 3.5.** In $\text{T}_0$, a map $e : A \to B$ is epimorphism if and only if for each $b \in B$, every neighborhood of $b$ intersects $\{b\} \cap e(A)$.

**Corollary 3.6.** If $B$ is a $\text{T}_1$, then the mapping $e$ is surjective.

Let $(X, T)$ be C-space and $\sim$ is a equivalence relation on $X$. Now we suppose $U = X/ \sim$ and $(U, V)$ is quotient topology w.r.t canonical function $q : X \to U$. Now we consider the quotient texture $(U, U)$ of $(X, T^c)$. It is easy to see that $V = U^c$.

We easily note here that $(U, U)$ is a quotient texture of $(S, S)$ iff $(U, U^c)$ is a C-space and $(U, U^c)$ is a quotient space of $(S, S^c)$.

By using the same argument in the proof of the above theorem we obtain that epimorphisms are precisely b-dense maps in $\text{CTop}$.

Equivalence functor reflects and preserves epimorphisms [14, Theorem 12.10]. Since $\text{Tex}$ is isomorphic(hence, equivalent) to $\text{CTop}$, we obtain following theorem:

**Theorem 3.7.** $e : (S, S) \to (T, T)$ is an epimorphism in $\text{Tex}$ if and only if $e : (S, S^c) \to (T, T^c)$ is an epimorphism in $\text{CTop}$.

From [10, Theorem 2.2.] we have:

**Corollary 3.8.** If $(T, T)$ is a discrete texture then $e$ is surjective.

In $\text{Top}$, extremal monomorphisms are topological embeddings and extremal epimorphism are the quotient functions. For the category $\text{Tex}$, we have:
Theorem 3.9. Let \( f : (S, S) \to (T, T) \) be a morphism in the category \( \text{Tex} \).

(a) Let \( f' : (S, S) \to (f(S), \mathcal{T}_{f(S)}) \) be defined by \( f'(s) = f(s) \), where \( s \in S \). Then:

\( f \) is extremal monomorphism \( \Rightarrow \) \( f' \) is an isomorphism.

(b) \( f \) is extremal epimorphism if and only if \( f \) is surjective and \( T \) is quotient texturing.

Proof. (a) We suppose \( f \) is extremal monomorphism. The following diagram commutative, where \( i : f(S) \to T \) is inclusion function.

\[
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\downarrow{f'} & & \downarrow{i} \\
& f(S) & \\
\end{array}
\]

Since \( f = i \circ f' \) is continuous and \( (f(S), \mathcal{T}_{f(S)}) \) is natural subtexture of \( (T, \mathcal{T}) \), \( f' \) is continuous. We note here that \( f'(S) = f(S) \), so \( f' \) is surjective. From the Proposition 2.5.(2) it is a epimorphism. Because \( f \) is extremal monomorphism, we have \( f' \) is an isomorphism.

(b)(\( \Rightarrow \)) We suppose \( f \) is extremal epimorphism. Let \( \sim_f \) be defined by \( x \sim_f y \iff f(x) = f(y) \). It is a equivalence relation on \( S \). Let \( \mathcal{U} \) be quotient texture on \( S/\sim_f \) w.r.t., the canonical function \( q : S \to S/\sim_f \). The following diagram commutative.

\[
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\downarrow{q} & & \downarrow{s} \\
S/\sim_f & & \\
\end{array}
\]

From the construction, \( s \) is injective function. Because \( s \circ q = f \) is continuous and \( S/\sim_f \) has quotient texturing, \( s \) is continuous. From the Proposition 2.5.(1), it is a monomorphism. Since \( f \) is extremal epimorphism, we have \( s \) is isomorphism. Hence, \( S/\sim_f \) and \( T \) have the same texturing.

(\( \Leftarrow \)) Suppose \( f \) is surjective and \( \mathcal{T} \) is quotient texturing. Let \( f = g \circ h \), where \( h : X \to Z \) and \( g : Z \to Y \) morphism in the category \( \text{Tex} \) and \( g \) monomorphism. Since \( f \) surjective, \( g \) is surjective and \( f \) is epimorphism. Hence \( g \) is bijective. So, there exists a function \( k : Y \to Z \) such that \( g \circ k = \iota_Y \) and \( k \circ g = \iota_Z \). Since \( k \circ f = k \circ g \circ h = \iota_Z \circ h = h \) and \( h \) is continuous, \( k \) is continuous. Then \( g \) is isomorphism. \( \square \)
4 The Category Tex is complete and cocomplete

Let \((S, S)\) be the product of the textures \((S_j, S_j)\) and \(\pi_j : S \to S_j\) projection functions, \(j \in J\), where \(J\) is a set.

Note here that \(E(j, A_j)\) is the preimage of \(A_j\) under the \(j\).th projection function \(\pi_j : S \to S_j\), i.e. \(E(j, A_j) = \pi_j^{-1}(A_j)\). So, the projection function \(\pi_j\) is continuous, for all \(j \in J\). Then:

**Theorem 4.1.** The pair \(((S, S), ((S, S) \xrightarrow{\pi_j} (S_j, S_j))_{j \in J})\) is a product of the family \((S_j, S_j)_{j \in J}\) in \(\text{Tex}\).

**Proof.** Take \((Z, Z) \in \text{Ob Tex}\) and \(p_j \in \text{Tex}((Z, Z), (S_j, S_j)), j \in J\). We must establish the existence of a unique function \(p\), which makes the below diagram commutative for each \(j \in J\).

\[
\begin{array}{ccc}
Z & \xrightarrow{p} & S \\
\downarrow{p_j} & & \downarrow{\pi_j} \\
S_j & & \\
\end{array}
\]

Now we consider the function \(p : Z \to S\) such that \(p_j(x) = \pi_j(p(x)), j \in J, x \in Z\). \(p\) is also continuous, because \(\pi_j\) and \(p_j\) are continuous, for all \(j \in J\). Hence \(p \in \text{Mor Tex}\). Since projection functions are monomorphism, \(p\) is unique which makes above diagram commutative for each \(j \in J\).

**Corollary 4.2.** The category \(\text{Tex}\) has product.

Since the product of sober spaces is sober \([9]\), the product of simple textures is simple. We have:

**Corollary 4.3.** The category \(\text{STex}\) has product.

Since the product of plain spaces is plain, it may be shown that the categories \(\text{PTex}\) and \(\text{PSTex}\) also have products. The details are left to the interested reader.

Let \((S, S)\) be the sum of the textures \((S_j, S_j)\) and \(\mu_j : S_j \to S\) injection functions, \(j \in J\), where \(J\) is a set.

Note here that \(A \cap S_j\) is the preimage of \(A\) under the \(j\).th injection function \(\mu_j : S_j \to S\), i.e. \(A \cap S_j = \mu_j^{-1}(A)\). So, the injection function \(\mu_j\) is continuous, for all \(j \in J\). Then:
**Theorem 4.4.** The pair \(((S_j, S_j) \xrightarrow{\mu_j} (S, S))_{j \in J}, (S, S)\) is a coproduct of the family \((S_j, S_j)_{j \in J}\) in \(\text{Tex}\).

**Proof.** Take \((Z, Z) \in \text{Ob Tex}\) and \(e_j \in \text{Tex}((S_j, S_j), (Z, Z)), j \in J\). We must establish the existence of a unique function \(\mu\), which makes the below diagram commutative for each \(j \in J\).

\[
\begin{array}{ccc}
S_j & \xrightarrow{e_j} & Z \\
\downarrow{\mu_j} & & \downarrow{\mu} \\
S & \xrightarrow{\mu} & Z
\end{array}
\]

We know that there exists \(s_j \in S_j\) such that \(s = (s_j, j)\), for each \(s \in S\). Now we consider the function \(\mu : S \to Z\) such that \(\mu(s) = \mu(s_j, j) = \mu(\mu_j(s_j)) = e_j(s_j) \in Z\). \(\mu\) is also continuous because \(\mu_j\) and \(e_j\) are continuous, for all \(j \in J\). Hence \(\mu \in \text{Mor Tex}\). Since injection functions are epimorphism, \(\mu\) is unique which makes above diagram commutative for each \(j \in J\). \(\square\)

**Corollary 4.5.** The category \(\text{Tex}\) has coproduct.

The sum of simple (plain) textures is simple (plain). Then we have:

**Corollary 4.6.** The categories \(\text{STex}, \text{PTex}\) and \(\text{PSTex}\) have coproduct.

Let \((S, S) \in \text{Ob (Tex)}\). Then, from the Theorem 3.1. \((S, \mathcal{T}) \in \text{Ob (CSober)}, \) where \(\mathcal{T} = S^c\). Now we consider subspace \((A, \mathcal{T}_A)\) of \((S, \mathcal{T})\), where \(A \subseteq S\). So, \((A, \mathcal{T}_A)\) is a natural subtexture of \((S, S)\). We have:

**Theorem 4.7.** Let \(f, g : (S, S) \to (T, \mathcal{T})\) be \(\text{Tex}\) morphisms. Then:

(a) If the set \(K = \{s \in S \mid f(s) = g(s)\}\) considered as a naturally subtexture of \((S, S)\) and \(\iota : K \to S\) is the inclusion function, then the pair \((K, \iota)\) is an equalizer of \(f\) and \(g\).

(b) Let \(\sim\) be the smallest equivalence relation (resp. congruence) on \((T, \mathcal{T})\) that contains all pairs \((f(s), g(s))\) for \(s \in S\), let \(U = T/\sim\) with the induced structure and let \(q : T \to U\) be the induced canonical function. Then the pair \((q, U)\) is a coequalizer of \(f\) and \(g\).

**Proof.** (a) Since \(f\) and \(g\) is equal on \(K\), \(f \circ \iota = g \circ \iota\). Let \(h : (Z, Z) \to (S, S)\) be a morphism in \(\text{Tex}\) where \(f \circ h = g \circ h\). Now we consider \(h' : Z \to K\) such that \(h'(z) = h(z)\), for all \(z \in Z\). It makes the below diagram commutative.
We easily note here that \( h(z) \in K \). From the definiton of naturally subtexture \( h' \) is continuous. Since inclusion function is monomorphism, \( h' \) is unique which makes above diagram commutative.

(b) Since \( f(s) \sim g(s) \), \( q \circ f = q \circ g \), for all \( s \in S \). Let \( h : (T, T) \rightarrow (Z, Z) \) be a morphism in Tex where \( h \circ f = h \circ g \). Now we consider the relation \( \pi_h \) on \( T \) which is defined by \( t \pi h t' \iff h(t) = h(t') \). Since \( h(f(s)) = h(g(s)) \), \( \pi_h \) is equivalent relation for all \( s \in S \). We note easily that \( \sim \subseteq \pi_h \) because \( \sim \) is smallest equivalent relation on \( T \). Hence there exists a function \( \text{id}_T^* : T/\sim \rightarrow T/\pi_h \) which makes the below diagram commutative.

\[
\begin{array}{cccc}
T & \xrightarrow{id_T} & T & \xrightarrow{h} & Z \\
q_1 & & q_2 & & \\
T/\sim & \xrightarrow{id_T^*} & T/\pi_h & & \\
\end{array}
\]

Now let \( s \circ \text{id}_T^* \circ h' = h \). So we obtain

\[ h' \circ q_1 = (s \circ \text{id}_T^* \circ q_1) = s \circ (\text{id}_T^* \circ q_1) = s \circ (q_2 \circ \text{id}_T) = (s \circ q_2) \circ \text{id}_T = h \circ \text{id}_T = h. \]

Because \( h \) continuous and \( q_1 \) is canonical function, \( h' \) is also continuous. Hence \( h' \in \text{Mor} \text{Tex} \). Since canonical function is epimorphism, \( h' \) is unique which makes above diagram commutative.

\[ \square \]

If any category has product(coproduct) and equalizer (coequalizer), then it is called complete (cocomplete) category [14, Theorem 23.8]. From Theorems 4.1, 4.4. and 4.7, we have:

**Corollary 4.8.** The category Tex is complete and cocomplete.

Every equalizer is extremal monomorphism and every coequalizer is extremal epimorphism. Then:

**Corollary 4.9.**

1. Tex is (epi,extremal-mono) factorizable.

2. Tex is (extremal-epi,mono) factorizable.
References


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