Almost Periodic Solutions of Some Nonlinear Fractional Differential Equations

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Abstract

In this paper, which is continuation of [1], As in [5, 10], we use the theory of fractional calculus to establish the existence and uniqueness of almost periodic solutions of a class of nonlinear fractional differential equations with analytic semigroup in Banach space, and we prove under suitable conditions that their optimal mild solutions are also weakly almost periodic.

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1 Introduction

Many dynamical systems are represented by the following nonlinear fractional differential equations:

\[
\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f(t, u(t)), t > t_0
\]  

(1.1)

in Banach space \(X\), where \(0 < \alpha \leq 1, t \geq 0\), we assume that \(-A\) is the infinitesimal generator of an analytic \(c_0\)-semigroup \(Q(t)\) satisfying the exponential stability, \(f\) is uniformly almost periodic function defined on \(R \times X_q\) into \(X\) satisfies the hypothesis \(F\): There are numbers \(L \geq 0\) and \(0 \leq \eta \leq 1\) such that

\[
|f(t_1, u_1) - f(t_2, u_2)| \leq L(|t_1 - t_2|^\eta + |u_1 - u_2|_q),
\]
for all \((t_1, u_2), (t_2, u_2)\) in \(R \times X_q\), where \(X\) is a real or complex Banach space with norm \(|.|\), \(A^q\) is the fractional power and \(X_q\) is the Banach space \(D(A^q)\) endowed with the norm \(|u|_q = |A^q u|\).

M.Bahaj and O.Sidki [10] proved the existence and uniqueness of almost periodic solution of (1.1) with conventional derivatives. G. N’Guerekata [5] gave necessary conditions to ensure that the so-called optimal mild solutions of \(u'(t) = Au(t) + f(t)\) are weakly almost periodic. As new in this paper we are concerned with fractional order.

In section 2, we state the basic notations, definitions and properties which are used throughout this work to obtain our results. In section 3, we establish the existence and uniqueness of almost periodic solution over \(R\) of (1.1). In section 4, we prove again the existence and uniqueness of the optimal mild solution of (1.1). In section 5, we show under necessary conditions that the optimal mild solution is also weakly almost periodic.

2 Preliminaries

Let \(X\) denote a real or complex Banach space endowed with the norm \(|.|\) and by \(\mathcal{L}(X)\) stands for the Banach algebra of bounded linear operators defined on \(X\). For \(A\) a linear operator with domain \(D(A)\), we denote by \(\mathcal{R}(A)\) the range of \(A\).

Following Gelfand and Shilov, we define the fractional integral of order \(\alpha > 0\) as

\[
I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,
\]

also, the fractional derivative of the function \(f\) of order \(0 < \alpha < 1\) as

\[
\frac{d}{dt} D_0^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(s)(t-s)^{-\alpha} ds,
\]

where \(f\) is an abstract continuous function on the interval \([a, b]\) and \(\Gamma(\alpha)\) is the Gamma function, see [1, 7].

Let \(-A\) is the infinitesimal generator of an analytic semigroup in a Banach space and \(0 \in \rho(A)\), \(\rho(A)\) is the resolvent set of \(A\). We define the fractional power \(A^{-q}\) by

\[
A^{-q} = \frac{1}{\Gamma(q)} \int_0^\infty (t)^{q-1} \mu(t) dt, q > 0.
\]

For \(0 < q \leq 1\), \(A^q\) is a closed linear operator whose domain \(D(A^q) \supset D(A)\) is dense in \(X\), this implies that \(D(A^q)\) endowed with the graph norm

\[
|u|_{D(A)} = |u| + |A^q u|, u \in D(A^q)
\]

is a Banach space, clearly \(A^q = (A^{-q})^{-1}\) because \(A^{-q}\) is one to one. Since \(0 \in \rho(A)\), \(A^q\) is invertible, and its graph norm is equivalent to the norm
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|u|_q = |A^q u|. Thus D(A^q) equipped with the norm |.|_q is a Banach space denoted by X_q, for more details we refer to [2,3].

**Lemma 2.1** Let −A be the infinitesimal generator of an analytic semigroup Q(t).

If 0 ∈ ρ(A) then
(a) Q(t) : X → D(A^q) for every t > 0 and q ≥ 0
(b) For every u ∈ D(A^q), we have Q(t)A^q u = A^q Q(t)u
(c) For every t > 0 the operator A^q Q(t) is bounded and |A^q Q(t)|_{L(X)} ≤ M_q t^{-q} e^{-δt}
(d) For 0 < q ≤ 1 and u ∈ D(A^q), we have |Q(t)u - u| ≤ C_q t^q |A^q u|, see [2, section 2.6].

The theory of almost periodic functions with values in a Banach space was developed by H. Bohr, S. Bochner, J. von Neumann, and others; cf., e.g., [9,18]. From their results (in addition, some techniques in [5,6,10,16,23]), we will mention several results which will be used in this work.

Let C_b(R,X) denote the usual Banach space of bounded continuous functions from R into X under the supremum norm |.|∞. Given a function f : R → X and ω ∈ R, we define the ω−translate of f as f_ω(t) = f(t + ω), t ∈ R. We will denote by H(f) = {f_ω : ω ∈ R} the set of all translates of f.

**Definition 2.1** (Bochner’s characterization of almost periodicity)
A function f ∈ C_b(R,X) is said to be almost periodic if and only if H(f) is relatively compact in C_b(R,X). We note that almost periodic functions can as well be characterized in terms of relatively dense sets in R of τ−almost periods.

**Definition 2.2** A function f : R → X is called almost periodic if
(i) f is continuous, and
(ii) for each ε > 0 there exists l(ε) > 0 , such that every interval I of length l(ε) contains a number τ such that |f(t + τ) - f(t)| < ε for all t ∈ R.

Let Y denote a Banach space and Ω an open subset of Y.

**Definition 2.3** A continuous function f : R × Ω → X is called uniformly almost periodic if for every ε > 0 and every compact set K ⊂ Ω there exists a relatively dense set P_ε in R such that |f(t + τ, u) - f(t, u)| ≤ ε for all t ∈ R, τ ∈ P_ε and all u ∈ K.

More details about this definition can be found in[24, P.188].

**Lemma 2.2** Let f : R × Ω → X be uniformly almost periodic and u : R → Ω be an almost periodic function such that R(u) ⊂ Ω, then the function t → f(t, u(t)) also is almost periodic. The proof in [22, Theorem I.2.7].

### 3 Almost Periodic Solutions

By a classical solution of (1.1) on [0,T), we mean a function u with values in X such that:
1) $u$ is continuous function on $[0, T)$ and $u(t) \in D(A)$,
2) $\frac{du}{dt}$ exists and continuous on $(0, T)$, $0 < \alpha < 1$, and $u$ satisfies (1.1) on $(0, T)$.

It is suitable to rewrite equation (1.1) in the form

$$ u(t) = u(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} [-Au(s) + f(s, u(s))]ds. \quad (3.1) $$

According to [1, 10-14], a solution of equation (3.1) can be formally represented by

$$ u(t) = \int_{0}^{\infty} \zeta_\alpha(\theta)Q((t - t_0)^\alpha \theta)u(t_0)d\theta + \alpha \int_{t_0}^{t} \int_{0}^{\infty} \theta(t - s)^{\alpha - 1}\zeta_\alpha(\theta)Q((t - s)^\alpha \theta)f(s, u(s))d\theta ds, \quad (3.2) $$

where $\zeta_\alpha$ is a probability density function defined on $(0, \infty)$ such that its Laplace transform is given by

$$ \int_{0}^{\infty} e^{-\theta x} \zeta_\alpha(\theta)d\theta = \sum_{j=0}^{\infty} \frac{(-x)^j}{\Gamma(1 + \alpha j)}, 0 < \alpha \leq 1, x > 0, \quad (3.3) $$

also, we have

$$ \int_{0}^{\infty} \theta^\eta \zeta_\alpha(\theta)d\theta \leq 1, 0 \leq \eta \leq 1, $$

for the proof of existence and uniqueness of solution of (3.2) we refer to [11, theorem 3.1 p.435], also as different method see [12, section 2, p. 824-827].

By a mild solution of (1.1), we mean a continuous solution of the integral equation (3.2).

When $A$ generates a semigroup with negative exponent, we deduce that if $u(.)$ is a bounded mild solution of (1.1) on $\mathbb{R}$, then we take the limit as $t_0 \to -\infty$ on the right-hand side of (3.2) and using (3.3), we obtain

$$ u(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t - s)^{\alpha - 1}\zeta_\alpha(\theta)Q((t - s)^\alpha \theta)f(s, u(s))d\theta ds. \quad (3.4) $$

Conversely, if $u(.)$ is a bounded continuous function and (3.4) is verified, then $u(.)$ is a mild solution of (1.1).

**Theorem 3.1** Let $-A$ be the infinitesimal generator of an analytic semigroup $\{Q(t)\}_{t \geq 0}$ satisfying $|Q(t)|_{L(X)} \leq Me^{\beta t}$, for all $t > 0$ and $\beta < 0$. If $f: \mathbb{R} \times X \to X$ is uniformly almost periodic and $f$ satisfies the assumption (F), then (1.1) has a unique almost periodic (classical) solution over $\mathbb{R}$ for $L$ sufficiently small enough.

For the proof, we shall need the following Lemma.
Lemma 3.2 If \( f : R \rightarrow X \) is almost periodic and locally Hölder continuous, then (1.1) has a unique almost periodic classical solution over \( R \) given by

\[
u(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \zeta_{\alpha}(\theta) Q((t-s)^{\alpha}\theta)g(s)d\theta ds.
\]

For the proof we can use the same technique which appear in Zaidman \([20]\). From Pazy \([3]\), clearly that if \( f : R \rightarrow X \) is Hölder continuous and if \( A \) generates an analytic semigroup, then the mild solution of (1.1) in fact is a classical solution.

We define the set \( AP(X) = \{ \varphi : R \rightarrow X, \varphi \) is almost periodic\} with the usual supremum norm over \( R \) which denoted by \( |.|_{\infty} \). We define on the set \( AP(X) \) a mapping

\[
T \varphi(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \zeta_{\alpha}(\theta) A^{q} Q((t-s)^{\alpha}\theta)f(s, A^{-q}\varphi(s))d\theta ds. \tag{3.5}
\]

We show that \( T \) is well defined. Let \( \varphi \in AP(X) \), using a standard properties of the almost-periodicity, we have \( N = \sup_{t \in R}|f(t, A^{-q}\varphi(t))| < \infty \). By Lemma 2.1.c, we have

\[
|T \varphi(t)| \leq \alpha NMq \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q}\zeta_{\alpha}(\theta)(t-s)^{-\alpha q+\alpha-1}e^{-\theta(t-s)^{\alpha}} d\theta ds.
\]

Set \( \eta = t - s \), we obtain

\[
|T \varphi(t)| \leq \alpha NMq \int_{0}^{\infty} \int_{0}^{\infty} \theta^{1-q}\zeta_{\alpha}(\theta)(\eta)^{-\alpha q+\alpha-1}e^{-\theta(\eta)^{\alpha}} d\theta d\eta.
\]

By using the properties of the probability density function \( \zeta_{\alpha} \), and the definition of the gamma function we conclude that \( T \varphi \) exists.

Lemma 3.3 The operator \( T \) is well defined, and maps \( AP(X) \) into itself.

Proof It follows from Lemma 2.2 that for \( \varphi \in AP(X) \), \( t \rightarrow f(t, A^{-q}\varphi(t)) \) is almost periodic. Hence, for each \( \epsilon > 0 \) there exists a set \( P_{\epsilon} \) relatively dense in \( R \) such that

\[
|f(t+\tau, A^{-q}\varphi(t+\tau)) - f(t, A^{-q}\varphi(t))| \leq \epsilon,
\]

for all \( t \in R \) and \( \tau \in P_{\epsilon} \). Therefore, the map \( T \) defined by (3.5) satisfies

\[
|T \varphi(t+\tau) - T \varphi(t)| = |\alpha \int_{-\infty}^{t+\tau} \int_{0}^{\infty} \theta(t+\tau-s)^{\alpha-1} \zeta_{\alpha}(\theta) A^{q} Q((t+\tau-s)^{\alpha}\theta)f(s, A^{-q}\varphi(s))d\theta ds

- \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \zeta_{\alpha}(\theta) A^{q} Q((t-s)^{\alpha}\theta)f(s, A^{-q}\varphi(s))d\theta ds|

= |\alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \zeta_{\alpha}(\theta) A^{q} Q((t-s)^{\alpha}\theta)f(s + \tau, A^{-q}\varphi(s + \tau))d\theta ds

- \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \zeta_{\alpha}(\theta) A^{q} Q((t-s)^{\alpha}\theta)f(s, A^{-q}\varphi(s))d\theta ds|

\leq \epsilon \alpha NMq \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q}\zeta_{\alpha}(\theta)(t-s)^{-\alpha q+\alpha-1}e^{-\theta(t-s)^{\alpha}} d\theta ds.
\]
Thus the function $T\varphi$ is almost periodic and $T : AP(X) \to AP(X)$.

**Proof of Theorem 3.1** Consider the mapping from the Banach space $AP(X)$ into itself defined by

$$T\varphi(t) = \psi(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \zeta_{\alpha}(\theta) A^{q} Q((t-s)^{\alpha}) f(s, A^{-q}\varphi(s)) d\theta ds.$$

Let $\varphi_{1}, \varphi_{2} \in AP(X)$, by using Lemma 2.1.c and assumption (F) we get

$$|T\varphi_{1} - T\varphi_{2}| \leq \alpha LM_{q} |\varphi_{1} - \varphi_{2}| \int_{-\infty}^{t} \int_{0}^{\infty} \theta^{1-q} \zeta_{\alpha}(\theta)(t-s)^{-\alpha q + \alpha - 1} e^{-\delta(t-s)^{\alpha}} d\theta ds,$$

again use the substitution $\eta = t - s$, we obtain

$$|T\varphi_{1} - T\varphi_{2}| \leq \alpha LM_{q} |\varphi_{1} - \varphi_{2}| \int_{-\infty}^{\infty} \theta^{1-q} \zeta_{\alpha}(\theta)(\eta)^{-\alpha q + \alpha - 1} e^{-\delta(\eta)^{\alpha}} d\theta d\eta,$$

It is known from above that the double integral in the right-hand side of the inequality exists, then we choose $L$ sufficiently small, thus $T$ is a strict contraction. By the contraction mapping theorem there exists $\varphi \in AP(X)$ such that

$$\varphi(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \zeta_{\alpha}(\theta) A^{q} Q((t-s)^{\alpha}) f(s, A^{-q}\varphi(s)) d\theta ds. \quad (3.6)$$

Since $A^{q}$ is closed, applying $A^{-q}$ on both sides of (3.6), we get

$$A^{-q} \varphi(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t-s)^{\alpha-1} \zeta_{\alpha}(\theta) Q((t-s)^{\alpha}) f(s, A^{-q}\varphi(s)) d\theta ds. \quad (3.7)$$

We show that the solution $\varphi$ of (3.7) is Hölder continuous on $R$. By Lemma 2.1.d, for every $\beta$ satisfying $0 < \beta < 1 - q$ and for every $h > 0$, we have

$$|(Q(h) - I)A^{q} Q(t - s)| \leq C_{\beta} h^{\beta} |A^{q+\beta} Q(t - s)|. \quad (3.8)$$

Also for $h \geq 0$, we can write

$$|Q((t + h - s)^{\alpha})| = |Q((t + h - s)^{\alpha} - (t - s)^{\alpha}h^{\alpha}\theta^{*} - h^{\alpha}\theta^{*}) Q(h^{\alpha}\theta^{*}) Q((t - s)^{\alpha})| \leq M^{*}|Q(h^{\alpha}\theta^{*}) Q((t - s)^{\alpha})|,$$

where $\theta^{*} = \frac{\delta}{2}$ (to ensure that $Q$ is defined) and $M^{*}$ is a constant. Using (3.8), (3.9) and Lemma 2.1.c, we get

$$|\varphi(t + h) - \varphi(t)| \leq |\alpha M^{*} \int_{-\infty}^{t} \int_{0}^{\infty} \theta[(t - s)^{\alpha-1} - (t - s)^{\alpha-1}] \zeta_{\alpha}(\theta) (Q(h^{\alpha}\theta^{*}) - I) A^{q} Q((t - s)^{\alpha}) f(s, A^{-q}\varphi(s)) d\theta ds|$$
Moreover, \( \psi \) is equipped with a norm \( \| \cdot \| \) on both sides of (3.13), we get
\[
\alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t + s - h) \zeta(\theta) d\theta d\eta
\]
\[
\leq \alpha M \| C \| h (t - s)^{\alpha \beta} \int_{-\infty}^{t} \int_{0}^{\infty} \left| \theta(t + s - h)^{\alpha - 1} \right| d\theta d\eta
\]
\[
+ \alpha N \| M \| \int_{0}^{t} \int_{0}^{\infty} \left| \theta(t + s - h)^{\alpha - 1} \right| d\theta d\eta.
\] (3.10)

We can estimate each term of the inequality separately to get \(| \varphi(t + h) - \varphi(t) | \leq C h^\beta \), which means that \( \varphi \) is Hölder continuous on \( R \). From assumption (F) we have
\[
| f(t, A^{-q} \varphi(t)) - f(s, A^{-q} \varphi(s)) | \leq L (| t - s |^\alpha + | \varphi(t) - \varphi(s) |).
\] (3.11)

Therefore \( t \rightarrow f(t, A^{-q} \varphi(t)) \) is Hölder continuous on \( R \). Let \( \varphi \) be the solution of (3.6) and consider the equation
\[
\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = f(t, A^{-q} \varphi(t)).
\] (3.12)

By Lemma 3.2, (3.12) has a unique solution given by
\[
\psi(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t + s - h)^{\alpha - 1} \zeta(\theta) Q((t - s)^{\alpha \theta}) f(s, A^{-q} \varphi(s)) d\theta d\eta.
\] (3.13)

Moreover, \( \psi(t) \in D(A) \subset D(A^q) \) for all \( t \in R \). Applying \( A^q \) on both sides of (3.13), we get
\[
A^q \psi(t) = \alpha \int_{-\infty}^{t} \int_{0}^{\infty} \theta(t + s - h)^{\alpha - 1} \zeta(\theta) A^q Q((t - s)^{\alpha \theta}) f(s, A^{-q} \varphi(s)) d\theta d\eta = \varphi(t).
\] (3.14)

Clearly \( \psi(t) = A^{-q} \varphi(t) \) is a solution of (1.1), the uniqueness of \( \psi \) follows from the uniqueness of the solution of (3.6) and (3.12). This completes the proof of the theorem.

In next sections, let us consider \( X \) be a uniformly convex Banach space equipped with a norm \( | \cdot | \) and \( X^* \) its topological dual space and \( \Omega \) an open subset of \( X \).

### 4 Optimal mild solutions

As in [1, 4, 5], we consider in \( X \) the equation (1.1) with the following assumptions:
F1: $A : D(A) \subset X \rightarrow X$ is a linear operator generates a $c_0$-semigroup of bounded linear operators $Q(t), t > 0$ satisfying $\sup_{t \in \mathbb{R}^+} |Q(t)| < \infty$.

F2: $f : \mathbb{R} \times \Omega \rightarrow X$ is a nontrivial strongly continuous function and is convex in $u$.

Let us denote by $\Omega_f$ the set of all mild solutions $u(t)$ of (1.1) which are bounded over $R$, that is

$$\mu(u) = \sup_{t \in \mathbb{R}} |u(t)| < \infty.$$  \hfill (4.1)

We assume here that $\Omega_f \neq \emptyset$, and we recall the following:

A bounded mild solution $\tilde{u}(t)$ of (1.1) is called an optimal mild solution of (1.1) if

$$\mu(\tilde{u}) \equiv \mu^* = \inf_{u \in \Omega_f} \mu(u).$$  \hfill (4.2)

**Theorem 4.1** Assume that $\Omega_f \neq \emptyset$ and the assumptions (F1-F2) are hold, then (1.1) has a unique optimal mild solution. (Compare with [21, theorem 1.1, p.138] and [5, theorem 1, p. 673]) Our proof is based on the following lemma.

**Lemma 4.2** If $K$ is a non-empty convex and closed subset of a uniformly convex Banach space $X$ and $v \notin K$, then there exists a unique $k_0 \in K$ such that $|v - k_0| = \inf_{k \in K} |v - k|$, see [17, Corollary 8.2.1].

**Proof of Theorem 4.1** It suffices to prove that $\Omega_f$ is a convex and closed set because the trivial solution $0 \notin \Omega_f$, then we use Lemma 4.2 to deduce the uniqueness of the optimal mild solution. For the convexity of $\Omega_f$, we consider two distinct bounded mild solutions $u_1(t)$ and $u_2(t)$, and a real number $0 \leq \lambda \leq 1$ and let $u(t) = \lambda u_1(t) + (1 - \lambda)u_2(t), t \in \mathbb{R}$. For every $t_0 \in \mathbb{R}$, $u(t)$ is continuous and (see [14]) has the integral representation

$$u(t) = T(t - t_0)u(t_0) + \int_{t_0}^{t} S(t - s)f(s, u(s))ds, t \geq t_0.$$  \hfill (4.3)

where

$$T(t) = \int_{0}^{\infty} \zeta_{\alpha}(\theta)Q(t^\alpha \theta)d\theta \& S(t) = \alpha \int_{0}^{\infty} \theta^{\alpha - 1}\zeta_{\alpha}(\theta)Q(t^\alpha \theta)d\theta.$$  

We have $u(t_0) = \lambda u_1(t_0) + (1 - \lambda)u_2(t_0)$ and $f(t, u)$ is convex in $u$, then $u(t)$ is a mild solution of (1.1). We note that $u(t)$ is bounded over $R$ since $\mu(u) = \sup_{t \in \mathbb{R}} |u(t)| \leq \lambda \mu(u_1) + (1 - \lambda)\mu(u_2) < \infty$, we conclude that $u(t) \in \Omega_f$.

Now we show that $\Omega_f$ is closed, let a sequence $u_n \in \Omega_f$ such that $\lim_{n \rightarrow \infty} u_n(t) = u(t), t \in \mathbb{R}$. For all $t_0 \in \mathbb{R}$ and $t \geq t_0$ we have

$$u_n(t) = T(t - t_0)u_n(t_0) + \int_{t_0}^{t} S(t - s)f(s, u_n(s))ds, \hfill (4.4)$$
It’s clearly that $T(t - t_0)$ and $S(t - s)$ are continuous operators, then for every fixed $t$ and $t_0$ with $t \geq t_0$, we have
\[
\lim_{n \to \infty} T(t - t_0)u_n(t_0) = \lim_{n \to \infty} \int_0^\infty \zeta_\alpha(\theta)Q((t - t_0)^\alpha \theta)u_n(t_0)d\theta
\]
\[
= \int_0^\infty \zeta_\alpha(\theta)Q((t - t_0)^\alpha \theta)d\theta \lim_{n \to \infty} u_n(t_0)
\]
\[
= T(t - t_0) \lim_{n \to \infty} u_n(t_0)
\]
\[
= T(t - t_0)u(t_0).
\]
Similarly we have
\[
\lim_{n \to \infty} \int_{t_0}^t S(t - s)f(s, u_n(s))ds = \int_{t_0}^t S(t - s) \lim_{n \to \infty} f(s, u_n(s))ds
\]
\[
= \int_{t_0}^t S(t - s)f(s, u(s))ds.
\]
Then we deduce that
\[
u(t) = T(t - t_0)u(t_0) + \int_{t_0}^t S(t - s)f(s, u(s))ds,
\]
for all $t_0 \in R, t \geq t_0$, which means that $u(t)$ is a mild solution of (1.1). Finally we show that $u(t)$ is bounded over $R$. We can write (4.3) as
\[
u(t) = T(t - t_0)u(t_0) + \int_{t_0}^t S(t - s)f(s, u(s))ds - u_n(t) + u_n(t)
\]
\[
= T(t - t_0)[u(t_0) - u_n(t_0)] + \int_{t_0}^t S(t - s)[f(s, u(s)) - f(s, u_n(s))]ds + u_n(t),
\]
for every $n = 1, 2, \ldots$, and every $t_0 \in R$ such that $t \geq t_0$.

Let $M = \sup_{t \in R^+} |Q(t^\alpha \theta)| < \infty$, since $\int_0^\infty \zeta_\alpha(\theta)d\theta = 1$, then $|T(t)| \leq M$, again (see [15,p.54]) since $\int_0^\infty t\zeta_\alpha(\theta)d\theta = 1$, then $|S(t)| \leq t|1 \sup_{t \in R^+} |Q(t^\alpha \theta)| \leq M|t|^{\alpha - 1}$, by assumption (F), we have $|f(s, u(s)) - f(s, u_n(s))| \leq L|u(s) - u_n(s)|_q$. These estimates lead to
\[
|u(t)| \leq M|u(t_0) - u_n(t_0)| + \alpha ML \int_{t_0}^t |t - s|^{\alpha - 1}|u(s) - u_n(s)|_q ds + |u_n(t)|.
\]
Choose $n$ large enough, for every $\epsilon > 0$ we get
\[
|u(t)| \leq M\epsilon + \alpha ML\epsilon \int_{t_0}^t |t - s|^{\alpha - 1}ds + \mu(u_n),
\]
then we have $\mu(u) \leq \epsilon_1 + \epsilon_2 + \mu(u_n) < \infty$. Thus $u \in \Omega_f$. This completes the proof of the theorem.
Weak almost periodic solutions

In order to formulate a property of almost periodic functions, which is equivalent to Definition 2.3, we discuss the concept of normality of almost periodic functions. Namely, let \( f(t, u) \) be almost periodic in \( t \in \mathbb{R} \) uniformly for \( u \in K \), then for every sequence of real numbers \( (s'_n) \) there exists a subsequence \( (s_n) \) and a function \( g(t, u) \) such that

\[
f(t + s_n, u) \longrightarrow g(t, u)
\]

uniformly on \( R \times K \) as \( n \longrightarrow \infty \), where \( K \) is a compact set in \( \Omega \), see Hamaya [24, p.188]. It is well known [4, 21] that:

\( f: \mathbb{R} \times \Omega \rightarrow X \) is weakly almost periodic if for every sequence of real numbers \( (s'_n) \) there exists a subsequence \( (s_n) \) such that every \( (f(t + s_n, u)) \) is convergent in the weak sense, uniformly on \( R \times K \). In other words, for every \( u^* \in X^* \), the sequence \( \langle u^*, f(t + s_n, u) \rangle \) is uniformly convergent on \( R \times K \), where \( \langle ., . \rangle \) denotes duality \( \langle X^*, X \rangle \). For each \( Q(t), t \in R^+ \), \( Q^*(t) \) denotes the adjoint operator of \( Q(t) \).

**Theorem 5.1** Let \( f(t, u) \) be almost periodic and assume that \((F1-F2)\) are hold, also assume that \( f \in L^1(\mathbb{R} \times \Omega, X) \) and \( Q^*(t) \in L(X^*) \) for every \( t \in R^+ \), then the optimal mild solution of (1.1) is weakly almost periodic.

**Proof** Let us consider \( u(t) \) is the unique optimal mild solution of (1.1), by theorem 4.1

\[
u(t) = T(t - t_0)u(t_0) + \int_{t_0}^{t} S(t - s)f(s, u(s))ds,
\]

for all \( t_0 \in R, t \geq t_0 \). Let \( (s'_n) \) be an arbitrary sequence of real numbers. Since \( f \) is almost periodic, we can extract a subsequence \( (s_n) \subset (s'_n) \) such that \( \lim_{n \rightarrow \infty} f(t + s_n, u) = g(t, u) \) uniformly on \( R \times K \). We note that \( g(t, u) \) is also strongly continuous. For fixed \( t_0 \in R \), we can obtain a subsequence of \( (s_n) \), which again we will denote \( (s_n) \), such that

\[
\text{weak} - \lim_{n \rightarrow \infty} u(t_0 + s_n) = v_0 \in X.
\]

Since \( X \) is a reflexive Banach space, then the function

\[
y(t) = T(t - t_0)v_0 + \int_{t_0}^{t} S(t - s)g(s, u(s))ds,
\]

is strongly continuous. It is a mild solution of

\[
\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = g(t, u(t)), t \in R.
\]

We need the following lemma.

**Lemma 5.2** For each \( t \in R \), we have

\[
\text{weak} - \lim_{n \rightarrow \infty} u(t + s_n) = y(t).
\]
Proof We can write
\[ u(t + s_n) = T(t - t_0)u(t_0 + s_n) + \int_{t_0}^{t} S(t - s)f(s + s_n, u(s))ds, \]
for every \( n = 1, 2, \ldots \), (see for instance [19, p.721]) let \( u^* \in X^* \), we have
\[ < u^*, T(t - t_0)u(t_0 + s_n) > - < u^*, T(t - t_0)v_0 > = < T^*(t - t_0)u^*, u(t_0 + s_n) - v_0 >, \]
for every \( n = 1, 2, \ldots \), we deduce that the sequence \( (T(t - t_0)u(t_0 + s_n)) \) converges to \( T(t - t_0)v_0 \) in the weak sense. Also we have
\[
\int_{t_0}^{t} S(t - s)f(s + s_n, u(s))ds - \int_{t_0}^{t} S(t - s)g(s, u(s))ds \\
\leq | \int_{t_0}^{t} S(t - s)[f(s + s_n, u(s)) - g(s, u(s))]ds | \\
\leq \int_{t_0}^{t} |S(t - s)||f(s + s_n, u(s)) - g(s, u(s))|ds \\
\leq M\alpha \int_{t_0}^{t} |t - s|^{\alpha - 1}|f(s + s_n, u(s)) - g(s, u(s))|ds.
\]
This leads to
\[
\lim_{n \to \infty} \int_{t_0}^{t} S(t - s)f(s + s_n, u(s))ds = \int_{t_0}^{t} S(t - s)g(s, u(s))ds,
\]
in the strong sense, then consequently in the weak sense in \( X \). We need also:

**Lemma 5.3**

\[ \mu(y) = \mu(u) = \mu^*. \]

**Proof** Since \( u(t) \) is an optimal mild solution of (1.1), we have \( \mu^* = \mu(u) = \sup_{t \in \mathbb{R}} |u(t)| \).

Let \( u^* \in X^* \), then by lemma 5.2 we obtain
\[
\lim_{n \to \infty} < u^*, u(t + s_n) > = < u^*, y(t) >,
\]
for every \( t \in \mathbb{R} \). For each \( n = 1, 2, \ldots \), we have
\[ | < u^*, u(t + s_n) > | \leq |u^*||u(t + s_n)| \leq |u^*|\mu^*. \]
Therefore, \[ | < u^*, y(t) > | = |u^*|\mu^* \] for every \( t \in \mathbb{R} \), and consequently \( |y(t)| \leq \mu^* \) for every \( t \in \mathbb{R} \), so that \( \mu(y) < \mu^* \).

We suppose that \( \mu(y) < \mu^* \). Note that \( \lim_{n \to \infty} g(t - s_n, u) = f(t, u) \) uniformly on \( \mathbb{R} \times K \) because \( f(t, u) \) is almost periodic. Since \( X \) is a reflexive Banach space, we can extract from the sequence \( (s_n) \), a subsequence which we still denote \( (s_n) \) such that \( (y(t_0 - s_n)) \) is weakly convergent to \( z \in X \). We have
\[
\lim_{n \to \infty} y(t - s_n) = T(t - t_0)z + \int_{t_0}^{t} S(t - s)f(s, u(s))ds
\]
in the weak sense for every $t \in R$. Now we consider the function
\[
z(t) = T(t - t_0)z + \int_{t_0}^t S(t - s)f(s, u(s))ds.
\]
It is a bounded mild solution of equation (1.1). Similarly as above, we have $\mu(z) \leq \mu(y)$, therefore $\mu(z) < \mu^*$, which is absurd by definition of $\mu^*$. We need also the following:

Lemma 5.4
\[
\mu(y) = \inf_{v \in \Omega_g} \mu(v)
\]
i.e. $y(t)$ is an optimal mild solution of the equation
\[
\frac{d^\alpha u(t)}{dt^\alpha} + Au(t) = g(t, u(t)), t \in R. \quad (5.1)
\]

**Proof** By lemma 5.3, $y(t)$ is bounded over $R$. Also $y(t)$ is a mild solution of (5.1) which means $y(t) \in \Omega_g$. It remains to prove that $y(t)$ is optimal. Suppose it is not. Since $\Omega_g \neq \emptyset$, by theorem 4.1, there exists a unique optimal solution $v(t)$ of (5.1). We have $\mu(v) < \mu(y)$ and
\[
v(t) = T(t - t_0)v(t_0) + \int_{t_0}^t S(t - s)g(s, u(s))ds,
\]
for all $t_0 \in R, t \geq t_0$. We can find a subsequence $(s_{n_k}) \subset (s_n)$ such that
\[
\lim_{k \to \infty} v(t - s_{n_k}) = T(t - t_0)z + \int_{t_0}^t S(t - s)f(s, u(s))ds \equiv V(t).
\]
Noting that $V(t) \in \Omega_f$ and $\mu(V) \leq \mu(v) < \mu(y)$, which is absurd. Therefore $y(t)$ is an optimal mild solution of (5.1), and in fact the only one by theorem 4.1.

**Proof of Theorem 5.1** To prove that $u(t)$ is weakly almost periodic, it suffices to show that
\[
weak \lim_{n \to \infty} u(t + s_n) = y(t)
\]
uniformly in $t \in R$. Suppose that this does not hold true; then there exists $u^* \in X^*$ such that
\[
\lim_{n \to \infty} < u^*, u(t + s_n) > = < u^*, y(t) >
\]
is not uniform in $t \in R$. Consequently, we can find a number $\gamma > 0$, and a sequence $(t_k)$ with two subsequences $(s'_k)$ and $(s''_k)$ of $(s_n)$ such that
\[
| < u^*, u(t + s'_k) - u(t + s''_k) > | > \gamma \quad (5.2)
\]
for all $k = 1, 2, ...$
Again, let us extract two subsequences of $(s'_k)$ and $(s''_k)$ respectively, with the
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same notation, such that
\[
\lim_{k \to \infty} f(t + t_k + s'_k, u) = g_1(t, u), \text{ and } \lim_{k \to \infty} f(t + t_k + s''_k, u) = g_2(t, u)
\]
both uniformly on \( R \times K \), because \( f \) is almost periodic. As we did previously, we may obtain
\[
\text{weak } \lim_{k \to \infty} f(t + t_k + s'_k, u) = T(t - t_0)z_1 + \int_{t_0}^{t} S(t - s)g_1(s, u(s))ds \equiv y_1(t),
\]
\[
\text{weak } \lim_{k \to \infty} f(t + t_k + s''_k, u) = T(t - t_0)z_2 + \int_{t_0}^{t} S(t - s)g_2(s, u(s))ds \equiv y_2(t)
\]
for each \( t \in R \), where \( y_1(t) \) and \( y_2(t) \) are optimal mild solutions in \( \Omega_{g_1} \) and \( \Omega_{g_2} \), respectively. Since \( \lim_{k \to \infty} f(t + t_k + s_k, u) \) exists uniformly on \( R \times K \), and \( (s'_k), (s''_k) \) are two subsequences of \( (s_k) \), we will get
\[
\sup_{s \in R} |f(s + s'_k, u) - f(s + s''_k, u)| < \epsilon
\]
if \( k \geq k_0(\epsilon) \) and consequently
\[
\sup_{s \in R} |f(t + t_k + s'_k, u) - f(t + t_k + s''_k, u)| < \epsilon
\]
for \( k \geq k_0(\epsilon) \), which shows that \( g_1(s, u(s)) = g_2(s, u(s)) \) for all \( s \in R \). By the uniqueness of the optimal mild solution we get \( y_1(t) = y_2(t), t \in R \). But \( y_1(0) = \text{weak } \lim_{k \to \infty} u(t_k + s'_k) \) and \( y_2(0) = \text{weak } \lim_{k \to \infty} u(t_k + s''_k) \). Clearly \( y_1(0) = y_2(0) \) contradicts the inequality (5.2) above. This completes the proof of theorem.

References


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