Complete Linear Systems on Rational Surfaces

Jesús Adrián Cerda Rodríguez and Brenda Leticia De la Rosa Navarro

Abstract
We compute the dimension of complete linear systems on some smooth projective rational surfaces. Thus giving a partial answer to a question posed by Mumford. The base field of the varieties under consideration is algebraically closed of arbitrary characteristic.

Mathematics Subject Classification: Primary 14J26, Secondary 14F17, 14F05

Keywords: Cohomology groups, Complete linear system, Monoidal transformation, Picard group, Rational surface

1 Introduction

In 1966 Mumford suggested four basic lines of study, one of them is the so-called “The Problem of Riemann-Roch” and it consists of determining the dimension of any complete linear system on a given smooth projective surface defined over an algebraically closed base field of any characteristic, see [4, page 4]. Motivated by the work in [2], we give a partial answer to Mumford question in the realm of smooth projective rational surfaces. For the reader conveniences, we suggest to look at [3] for basic notation and notions and at [2] and the references therein for useful properties. Our result is:

Theorem 1.1. Let \( \mathcal{F} \) be an invertible sheaf on the surface \( X \) obtained as the blow up the projective plane \( \mathbb{P}^2 \) at the zero-dimensional subscheme \( S \) having as a support \( r \) distinct collinear points of multiplicity two, \( r \) is a non-negative integer. Then either \( h^0(X, \mathcal{F}) \) is less than or equal to one or equals to

\[
1 + \frac{1}{2}(\mathcal{F}'^2 - \mathcal{F}' \cdot \mathcal{K}),
\]
where $\mathcal{F}'$ is the mobile invertible sheaf associated to $\mathcal{F}$, and $\mathcal{K}$ is the canonical sheaf on $X$.

**Remark 1.2.** Let $\mathcal{F}$ be an invertible sheaf on a given smooth projective surface $Y$ ($Y$ may not be rational) and assume that $\mathcal{F}$ has at least two global sections. Then $h^0(Y, \mathcal{F}) = h^0(Y, \mathcal{G})$, where $\mathcal{G}$ is either the nef invertible sheaf or the mobile invertible sheaf $\mathcal{F}'$ associated to $\mathcal{F}$.

The following well-known result is thus obtained (see [2, Remark 1.5]):

**Corollary 1.3.** Let $\mathcal{F}$ be an invertible sheaf on the surface $Z$ obtained as the blow up the projective plane $\mathbb{P}^2$ at the $r$ distinct collinear points, $r$ is a non-negative integer. Then either $h^0(Z, \mathcal{F})$ is less than or equal to one or equals to

$$1 + \frac{1}{2}(\mathcal{F}'^2 - \mathcal{F}.\mathcal{K}),$$

where $\mathcal{F}'$ is the mobile invertible sheaf associated to $\mathcal{F}$, and $\mathcal{K}$ is the canonical sheaf on $Z$.

Another consequence of Theorem 1.1 is the following:

**Corollary 1.4.** With notation as above. Let $D$ be an integral curve on the surface $X$. Then the dimension of global sections of the invertible sheaf $\mathcal{O}_X(D)$ is given by:

$$h^0(X, \mathcal{O}_X(D)) = \begin{cases} 
1 & \text{if } D^2 \text{ is less than zero}, \\
1 + \frac{1}{2}(D^2 - D.\mathcal{K}) & \text{if } D^2 \text{ is greater than zero}.
\end{cases}$$

Here $\mathcal{K}$ denotes a canonical divisor on $X$.

In particular, we have:

**Corollary 1.5.** With notation as above. The the dimension of global sections of the invertible sheaf $\mathcal{O}_X(D)$ is greater than two for any given integral curve $D$ on $X$ of self-intersection greater than or equal to one.

**Remark 1.6.** Other new families of smooth projective rational surfaces called “Platonic rational surfaces” for which the result of Theorem 1.1 holds can be found in the work [1].

2 Proof of Theorem 1.1

This section is devoted to give a short proof of the result announced in Theorem 1.1. With notation as in the theorem, we may at once assume that $\mathcal{F}$ has at least two global sections, and even more $\mathcal{F}$ is equal to its associated mobile invertible sheaf $\mathcal{F}'$. It follows then that:
\[ h^0(X, \mathcal{F}) = 1 + \frac{1}{2}(\mathcal{F}^2 - \mathcal{F}^\prime, \mathcal{K}) + h^1(X, \mathcal{F}^\prime). \]

Hence we need only to prove that \( h^1(X, \mathcal{F}^\prime) = 0 \). For, let us define the standard basis of the Picard group \( \text{Pic}(X) \) of \( X \) as follows:

\[ (\mathcal{E}_0, -\mathcal{E}_1, \ldots, -\mathcal{E}_r), \]

where

- \( \mathcal{E}_0 \) is the class of a general line in the projective plane.
- For every \( i = 1, \ldots, r \), \( \mathcal{E}_i \) denotes the class of the exceptional divisor corresponding to the \( i^{th} \) point blown-up \( P_i \).

Here the support of \( S \) is equal to the set \( \{P_1, \ldots, P_r\} \).

Henceforth, the class of a divisor on \( X \) is given by an \((1 + r)\)-tuple \((d, m_1, \ldots, m_r)\) of integers.

On the other hand, the Picard group \( \text{Pic}(X) \) of \( X \) is endowed with an intersection pairing which is given by the three requirements:

1. \( \mathcal{E}_0^2 = -(\mathcal{E}_i)^2 = 1 \) for every \( i = 1, \ldots, r \).
2. \( \mathcal{E}_i, \mathcal{E}_j = 0 \) for every \( i, j = 1, \ldots, r \), with \( i \neq j \).
3. \( \mathcal{E}_i, \mathcal{E}_0 = 0 \) for every \( i = 1, \ldots, r \).

let \((d, m_1, \ldots, m_r)\) be the \((1 + r)\)-tuple of integers associated to \( \mathcal{F}^\prime \) in the Picard group \( \text{Pic}(X) \) of \( X \) relatively to the standard basis. Since the invertible sheaf \( \mathcal{F}^\prime \) is nef and the support of the zero-dimensional subscheme \( S \) is contained in a line, we deduce that the integers \( d^2 - \sum_{i=1}^r m_i^2 \) and \( d - \sum_{i=1}^r m_i \) are non-negative. If the integer \( d - \sum_{i=1}^r m_i \) is positive, we are done. Then we may assume that \( d - \sum_{i=1}^r m_i \) is equal to zero. In this later case, it follows that \( \mathcal{F}^\prime \) is equal to the structure sheaf \( \mathcal{O} \) of \( X \), hence the result follows (since the surface \( X \) is rational).

**ACKNOWLEDGEMENTS.** This research was partially supported by grants CIC-UMSNH (Proyecto 4.25), COECYT- FIFOECYT 2008 and PROMEP UMSNH-PTC-214.
Figure 1: Some integral curves of interest on the surface $X$.

References


Received: September, 2008