Bipartite Theory of Mixed Domination

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Abstract. We give the bipartite theory of vertex-edge weak dominating set and edge-vertex strong dominating set of a graph.

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1. Introduction

A graph \( G=(V,E) \) is bipartite if its vertices can be partitioned into two subsets \( V_1 \) and \( V_2 \) such that for every edge \( (u,v) \in E \), either \( u \in V_1 \) and \( v \in V_2 \) or \( u \in V_2 \) and \( v \in V_1 \). If a connected graph \( G=(V,E) \) is bipartite, we could denote it \( G=(V_1,V_2,E) \) to indicate the bipartition of its vertices. Let \( G=(X,Y,E) \) denote an arbitrary bipartite graph.

Given an arbitrary graph \( G=(V,E) \), one can construct a variety of bipartite graphs \( G'=(X,Y,E') \) which faithfully represent \( G \), in the sense that given two graphs \( G \) and \( H \), \( G \) is isomorphic to \( H \) if and only if the corresponding bipartite graphs \( G' \) and \( H' \) are isomorphic. Here we give two such bipartite constructions.
Consider the graph $G$ in figure 1, whose vertices and edges are labeled. Figure 2 and figure 3 are two bipartite graphs, denoted by $VNe(G)$ and $EN^+(G)$ which can be constructed from $G = (V,E)$. The neighbor of an edge 1 is denoted by $N[1]$ is defined as $N[1] = N(a) \cup N(b)$, where a and b are vertices incident with edge 1. The graph $VNe(G) = (V,N[e],E')$ is defined by the edges $E' = \{(x,N[y]) / x \in N[y] \text{ in } G\}$.

The graph $EN^+(G) = (E,N[v],F)$ is given in Figure 1 the edges $F = \{(e,N[v]) / \text{ the vertices incident with } e \text{ are in } N[v]\}$.

[3] and [4] suggests that given any problem, say P, on an arbitrary graph G, there is very likely a corresponding problem Q on a bipartite graph $G^1$, such that a solution for Q provides a solution for P. Here we give the bipartite version of vertex-edge weak dominating set and edge-vertex strong dominating set.
2. Definitions

The definitions are given in [3] and [4]. Let \( G = (X, Y, E) \) be a bipartite graph. Two vertices \( u, v \in X \) are X-adjacent if they are adjacent to a common vertex in Y. A subset \( D \) of \( X \) is a Y-dominating set if every \( y \in Y \) is adjacent to at least one vertex in \( D \). The minimum cardinality of a Y-dominating set is called the Y-domination number and is denoted by \( \gamma_Y(G) \). A subset \( D \) of \( X \) is an X-dominating set if every \( x \in X - D \) is X-adjacent to at least one vertex \( u \in D \). The minimum cardinality of a X-dominating set is called the X-domination number and is denoted by \( \gamma_X(G) \).

The definitions are defined as in [1]. Let \( G^1 = (V, E) \) be a graph. \( v \) and \( x = uv \in E \) weakly dominate each other if \( v \in N[x] \). Vertex-edge weak domination number \( \gamma_{01}(G^1) \) of a graph \( G^1 \) is the minimum cardinality of a set of vertices weakly dominating all edges of \( G^1 \).

\( v \) and \( x = uv \in E \) strongly dominate each other if \( x \in \langle N[v] \rangle \). The edge-vertex strong domination number \( \gamma_{10}(G^1) \) is the minimum number of set of edges strongly dominates all vertices of \( G^1 \).

3. Main Result

**Theorem 1:** For a graph \( G \), \( \gamma_Y(VNe) = \gamma_{01}(G) \).
Proof: Let \( S \subseteq V \) be a \( \gamma_Y(VNe) \) set. Elements of \( S \) are adjacent to \( N[e] \forall e \in E(G) \).
In \( G \), elements of \( S \) weakly dominates edges of \( G \). Therefore, \( S \) is a vertex-edge weak dominating set. \( \gamma_0(G) \leq |S| = \gamma_1(VNe) \).
Conversely, \( S \) be a vertex edge weak dominating set. Elements of \( S \) weakly dominate all edges of \( G \). Equivalently for every edge \( x \in E(G) \), there exists \( \nu \in S \) such that \( \nu \in N[x] \). Elements of \( S \) is adjacent to at least one element of \( N[x] \), in graph \( VNe \). Therefore, \( S \) is a \( Y \)-dominating set in \( VNe \), \( \gamma_Y(VNe) \leq |S| = \gamma_01(G) \). Hence, \( \gamma_Y(VNe) = \gamma_01(G) \).

**Theorem 2:** Every distance 2-dominating set in \( G \) is a \( X \)-dominating set in \( VNe \).

**Proof:** Let \( S \) be a \( \gamma_2(G) \) set. \( \forall u \in V - S \exists v \in S \), such that \( u \) and \( v \) are at a distance 2.
Case (i): \( d(u, v) = 1 \).
\( u \) and \( v \) are incident to a common edge \( e \). In graph \( VNe \), \( u \) and \( v \) are incident to a vertex \( N[e] \). Hence, \( S \) is a \( X \)-dominating set in \( VNe \).

Case (ii): \( d(u, v) = 2 \).
Let \( u - v \) path be \( u\varepsilon \nu w_{2}\varepsilon v \). \( N[e_1] = N(u) \cup N(w) \) and \( N[e_2] = N[w] \cup N(v) \). In graph \( VNe \), \( u \) and \( v \) are incident with \( N[e_1] \) and \( N[e_2] \). Hence, \( S \) is a \( X \)-dominating set in \( VNe \).

Observation: The converse of the above theorem need not be true. In \( G \), \( S = \{a\} \) is not a distance 2-dominating set but in \( VNe \) the set \( S = \{a\} \) is a \( X \)-dominating set.
Consider the graph \( G \),

![Diagram of the graph VNe](image-url)

The graph \( VNe \).
**Theorem 3:** For any graph $G$, $\gamma_Y(EN^+) = S\gamma_{10}(G)$.

**Proof:** Let $D$ be a $\gamma_Y(EN^+)$ set. Elements of $D$ are adjacent to elements $N[v]$. In $G$, $D$ strongly dominates all vertices of $G$. Therefore, $S\gamma_{10}(G) \leq |D| = \gamma_Y(EN^+)$. Conversely, let $D$ be a $S\gamma_{10}(G)$ set. Edges in $D$ strongly dominates all vertices of $G$. $\langle N[v]\rangle$ contains at least one edge of $D$ for every $v \in V(G)$. $D$ is a Y-dominating set in $EN^+$. Therefore, $\gamma_Y(EN^+) \leq |D| = S\gamma_{10}(G)$. Hence, $\gamma_Y(EN^+) = S\gamma_{10}(G)$. □

**Theorem 4:** For any graph $G$, $\gamma_X(EN^+)=\gamma_1(G)$.

**Proof:** Let $D$ be a $\gamma_X(EN^+)$ set. $\forall x \in E - D$ there exists $y \in D$ such that $x$ and $y$ are X-adjacent. In $G$, edges $x$ and $y$ belongs to $\langle N[a]\rangle$ for some $a \in V(G)$. Edges $x$ and $y$ are incident to a common vertex in $N[a]$. $D$ is an edge dominating set. $\gamma_1(G) \leq |S| = \gamma_X(EN^+)$. Conversely, $D$ be a $\gamma_1(G)$ set. $\forall x \in E - D$, there exists $y \in D$ such that $x$ and $y$ are adjacent. Suppose $x$ and $y$ are incident at $u \in V$. The graph $\langle N[u]\rangle$ contains $x$ and $y$. In graph $EN^+$, $x$ and $y$ are adjacent to $N[u]$. Hence, $D$ is a X-dominating set in $EN^+$. Therefore, $\gamma_X(EN^+) \leq |D| = \gamma_1(G)$. Hence, $\gamma_X(EN^+) = \gamma_1(G)$. □
References


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